MAT 495 WEEK 4: RECURRENCES

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Recurrence problems

A recurrence is any sequence wherein one of the terms is determined in some way from the previous terms. A simple example is $f_1 = 1, f_n = f_{n-1}+1$ for $n \geq 1$. Of course, it turns out that $f_n = n$ is the unique solution to this recurrence.

Tips for dealing with Recurrences:

- Try to reduce to a recurrence you already know how to solve
- Try small case, and identify a pattern.
- If the problem is not formulated as a recurrence problem, what sequence can you define to try and find a recurrence it satisfies?
- For counting problems of the type 'count some set of configurations S_n depending on an integer n', try to reduce the count of S_n to that of S_k for values of k that are smaller than n.

2.1. General Theory: Linear Recurrences.

One very famous type of recurrence is a *linear recurrence*, a sequence satisfying a relation of the form $f_{n+m} = \sum_{i=0}^{m} c_i f_{n+i}$ for all n. An example is the Fiobonacci numbers, which satisfy $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for all natural numbers n. Here is how such recurrences can be solved:

The trick with linear recurrences is to look for solutions of the form c^n, for some constant c. Consider the recurrence $f_{n+2} = 3f_{n+1} - 2f_n$. If we try $f_n = c^n$, we end up with $c^{n+2} - 3c^{n+1} + c^n = 0$, or in other words $c^2 - 3c + 1 = 0$, which has the solutions $c = 1$ and $c = 2$. Thus, we see that 2^n, and $1^n (= 1)$ are both solutions to this recurrence. More generally, any linear combination is also a solution, and so $a2^n + b$ is a solution to our linear recurrence. In fact these are all the solutions (can you see why?)

To apply this method in general, just find all the roots of the polynomial $x^{n+m} - \sum_{i=0}^m c_i x^i$. Something a little different happens when you have repeated roots. Consider for example the recurrence $f_{n+2} - 4f_{n+1} + f_n = 0$. The resulting polynomial is $(x^2 - 4x + 4) = (x-2)^2$ and so the only exponential solution is $f_n = 2^n$. It turns out that in this case, $f_n = n2^n$ is also a solution. In general, if $(x-c)^k$ is a factor of your polynomial, then any of $c^n, nc^n, \ldots, n^{k-1}c^n$ will satisfy the recurrence(and so will linear combinations of them, of course!)

One word of warning: there is no reason for the roots of your polynomial to be rational! For example, prove that the n'th Fiboonacci number has the following formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

3. Problems

(1) Solve the linear recurrence $f_n = 2f_{n-1} + 1$, $f_1 = 1$.
(2) Solve $f_n = 5f_{n-1} - 6f_{n-2}$, $f_0 = 0, f_1 = 1$.
(3) Suppose $f_2 = 2$ and $f_n + 2f_{n-1} + 4f_{n-2} = 0$ for all n. Find f_{2015}.
(4) Suppose f_n is a sequence with $f_{n+3} = 2f_{n+2} + f_{n+1} - 2f_n$ for all $n \geq 0$, and $f_0 = 1, f_1 = 2, f_2 = 3$. Find f_{2015}.
(5) A sequence $a_0, a_1, \ldots, a_n, \ldots$ of real numbers is defined by $a_0 = 1$, and for $n \geq 0$,
$$a_{n+1} = \frac{a_n}{1 + na_n}.$$
Determine a_{2015}.
(6) A triangle has sides a, b, c. If possible, construct another triangle with sides $b+c-a, a+c-b, a+b-c$. For which triangles can this process be repeated infinitely many times?
(7) A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Find the probability of the student scoring exactly n points at some point in the sequence.
(8) (Puntam A1)Define a *selfish* set to be a set that has its own cardinality (number of elements) as an element. A *minimal selfish* set is a selfish set, none of whose proper subsets are selfish. I.e. $\{2, 3\}$ is minimal selfish, but $\{1, 2\}$ is not. Find, with proof, the number of subsets of $\{1, 2, 3, \ldots, n\}$ that are selfish.
(9) (Putnam 2013 B-1) For positive integers n, let $c(n)$ be determined by $c(1) = 1, c(2n) = c(n), c(2n+1) = -c(n)$. Find
$$\sum_{n=1}^{2015} c(n)c(n+2).$$

(10) (Putnam 1993 A-2)Let $x_n, n > 0$ be a sequence of non-zero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1.$$

Prove that there exists a real number a such that $x_{n+2} = ax_{n+1} - x_n$.

(11) Let F_m be the m'th Fiobonacci number, so that $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove that $F_m \mid F_n$ if and only if $m \mid n$.

(12) An elf skips up a flight of numbered stairs, starting at step 1 and going up one or two steps with each leap. He counts how many ways he can reach the n't step, calling the n'th number E_n. What is E_n?

(13) (Putnam 2006 A-3)Let $1, 2, 3, 4, \dots, 2014, 2015, 2017, 2020, 2024 \dots$ be a sequence defined by $x_k = k$ for $k = 1, 2 \dots, 2015$ and $x_{k+1} = x_k + x_{k-2013}$ for $k \geq 2014$. Show that the sequence eventually has 2013 consecutive terms each divisible by 2014.

(14) (Putnam 1998 A-4)Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, A_n is goten by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example, $A_3 = A_2A_1 = 10, A_4 = A_3A_2 = 101, A_5 = 10110$ and so on. Determine whether or not A_{2015} is divisible by 11.

(15) (Putnam 2000 A-6) A polynpmial $P(x)$ has integer coefficients. A sequence x_n is determined by $x_0 = 0$ and for positive $n, x_n = P(x_{n-1})$. Suppose $x_{2015} = 0$. Prove that $x_1 = 0$ or $x_2 = 0$.