MAT 495 WEEK 6: PROBABILITY

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Expectation

Say you have some sort of random process E, and for every possibly event $e \in E$ you have some number $X(e)$. For instance, you might be tossing coins, or picking real numbers at random. An example of what X might be the total number of heads if two fair coins are flipped. Then the random process is flipping two heads, and it has four events, HH, HT, TH, TT, and $X(TT) = 0, X(HT) = X(TH) = 1, X(HH) = 2$. Now, the expected value $E(X)$of X is defined to be the average of X over all events, weighted by the probability of the event. Formally speaking, $E(X) = \sum_{e \in E} P(e)X(e)$, where $P(e)$ is the probability that e will occur. So in our case, $E(X) = \frac{1}{4} \cdot (0 + 1 + 1 + 2) = 1$.

Informally speaking, if you generate events e at random for a very long time, and compute $X(e)$ each time, the average of all $X(e)$ you see should be very close to $E(X)$.

Here are some problems to get you started:

- I've just heard a public announcement that 80% of Utoronto students don't drink (to oblivion), 70% don?t get lazy(on a regular basis), and 51% don't party (every night). Prove mathematically that at least one Utoronto student came here to study.
- A fair coin (one which lands on heads half the time, and on tails half the time) is tossed 3 times. Prove that it is 50% likely to land heads an odd number of times.
- Slips of paper with the numbers from 1 to 99 are placed in a hat. Five numbers are randomly drawn out of the hat one at a time (without

replacement). What is the probability that the numbers are chosen in increasing order?

- One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?'

2.1. **Probability Problems.**

(1) The buses in Toronto going to the latest Blue Jays game were really full. Any bus that had more than 20 people riding in it was overfilled. Prove that the percentage of overfilled buses was at most as large as the percentage of passengers that were riding an overfilled bus.

(2) Choose, at random, three points on the circle $x^2 + y^2 = 1$. Interpret these points as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains the point $(1, 0)$.

(3) three points A, B, C are chosen uniformly at random on a circle. What is the probability that the center of the circle lies in the triangle ABC?

(4) You have $n > 1$ numbers $0, 1, \ldots, n-1$ arranged on a circle. A random walker starts at 0 and at each step moves at random to one of its two nearest neighbors. For each i, compute the probability p_i that, when the walker is at i for the first time, all other points have been previously visited, i.e., that i is the last new point. For example, $p_0 = 0$.

(5) Choose two random numbers from the closed interval $[0, 1]$ and let them be the endpoints of an interval. Repeat this n times, so that you end up with n intervals, which may or may not intersect. What is the probability that there is an interval which intersects all others?

(6) Start with n strings, which have $2n$ ends. Then randomly pair the ends together and tie each pair together in a knot. The result is L loops. Prove that the expected value of L is

$$\sum_{i=1}^{n} \frac{1}{2i-1} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}.$$

(7) Three players are in a room and a red or blue hat is placed on each person's head. The color of each hat is determined by [an independent] coin toss. No communication of any sort is allowed, but the players are allowed an initial strategy session to decide on a mutual strategy before the game begins. Once they have had a chance to look at the other hats [but not their own], the players must simultaneously guess the color of their own hats or pass. The players win if no-one guesses incorrectly, and at least one person does not pass. Find a strategy that maximizes their odds of winning.

(8) Let k be a positive integer. Suppose that the integers $1, 2, 3, ..., 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3?

(9) Let p_n be the probability that $c + d$ is a perfect square, where the integers c, d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Find the limit of $p_n\sqrt{n}$ as n tends to infinity.

(10) An $m \times n$ checkerboard is colored randomly: each square is assigned a color of red or blue with probability $1/2$. We say that two squares p, q are in the same monochromatic component if there exists a sequence of squares, starting with p and ending with q , all of the same color, in which successive squares share a common side. Prove the expected number of monochromatic components is at least $mn/16$. **Harder: at least mn/8**.