MAT 495 WEEK 1: PIGEONHOLE PRINCIPLE AND MATHEMATICAL INDUCTION

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. Plug in smaller numbers. Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Mathematical Induction

Suppose that $P(n)$ is a statement for each positive integer n. The principle of mathematical induction states that if

- $P(1)$ is true - this is usually called the *base case*.
- For each positive integer n, if $P(n)$ is true, then $P(n+1)$ is true as well,

then $P(n)$ is true for all positive integers n.

2.1. Problems.

(1) Show that $1 + 2 + \cdots + n = n(n+1)/2$.
(2) Show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.
(3) Find a formula for the sum of the first n odd numbers.
(4) Prove that for $n \geq 12$, then n cents of postage can be made up using only 7 cent and 3 cent stamps.
(5) Prove that for $n \geq 6$ a square can be dissected into n squares, not necessarily all of the same size.
(6) Let r be a real number such that $r + \frac{1}{r}$ is an integer. Prove that $r^{100} + \frac{1}{r^{100}}$ is also an integer.
(7) Prove the (very useful) principle of **strong induction**, which goes as follows:

    Suppose that $P(n)$ is a statement for each positive integer n. The principle of mathematical induction states that if

- $P(1)$ is true - this is usually called the *base case*.
- For each positive integer n, if $P(1), P(2), \ldots, P(n)$ are all true, then $P(n+1)$ is true as well,

then $P(n)$ is true for all positive integers n.

(8) The Fiobonacci sequene F_n is defined as follows. $F_1 = 1, F_2 = 1$ and for $n > 0$, $F_{n+2} = F_{n+1} + F_n$. So the sequences starts $1, 1, 2, 3, 5, 8, 13$. Prove that for any positive integer n, $F_n^2 + F_{n+1}^2 = F_{2n+1}$.

(9) For any set T whose elements are positive integers, define $f(T)$ to be the square of the product of the elements of T. For example, if $T = \{1, 2, 5\}$ then $f(T) = (1 \cdot 2 \cdot 5)^2 = 100$. For any positive integer n, consider all non-empty subsets S of $\{1, 2, \ldots, n\}$ that do not contain two consecutive positive integers. Prove that if one sums $f(S)$ over all of these subsets, one gets $(n+1)! - 1$.

(10) Prove the arithmetic mean - geometric mean inequality (AM-GM): Suppose $a_1, \ldots a_n$ are n positive real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{\frac{1}{n}}.$$

(a) First do the case of $n = 2$.
(b) Then do the case where n is a power of 2.
(c) Prove that if the statement is true for n then it is true for $n-1$.
(d) Prove that if the statement is true for n then it is true for $2n$.
(e) Conclude!

(11) We call a property *wacky* of the positive integers as follows: 1 and 2 are wacky by definition. For any positive integer n thats at least 3, we say that n is wacky if there is a unique way to write n as a sum of two distinct wacky numbers smaller thatn n. So 3 is wacky, 4 is wacky, 5 is not wacky, since 5=1+4=2+3, 6 is wacky, etc...

Prove that there are infinitely many wacky numbers.

(12) given a sequence $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$. Prove that for any positive integer k, the sequence $a_1, a_2, \ldots$ becomes constant modulo k. *for example, the sequence starts* $1, 2, 4, 16, 65536$ *which modulo* 3 *is* $1, 2, 1, 1, 1,$ *and it continues being* 1 *modulo* 3.

(13) Every road in the country Graphsville is one way, and every pair of cities is connected by exactly one road. Show that there is a city which can be reached by every other city either directly or by going through at most one other city.

3. The Pigeonhole Principle

The Pigeonhole principle is very simple: Given $kn+1$ pigeons distributed among n pigeonholes, there is a box with at least $k+1$ pigeons inside it.

To apply this principle, you often have to be very creative in deciding what are your pigeons, and what are your pigeonholes!

3.1. **Problems.**

(1) Consider 5 points inside a square of side length 1. Show that two of these points are most $\frac{1}{\sqrt{2}}$ apart.

(2) Choose any nine positive integers from 1 to 200. Prove that among your nine numbers, it is possible to choose two of them whose ratio lies between 1 and 2.

(3) (a) Show that the decimal representation of a positive rational number must eventually repeat.
   (b) Show that the decimal of a representation of a positive irrational number never repeats.

(4) Prove that in any group of six people there are either three mutual friends or three mutual strangers.

(5) Let N be a positive integer. Prove that some multiple of N (written in base ten, of course) consists entirely of 0's and 1's.

(6) Seventeen people correspond by mail with one another ? each one with all the rest. In either letters only three topics are discussed. Each pair of correspondents deals with only one of the topics. Prove that there are at least three people who write to each other about the same topic.

(7) (Putnam 2000B1) Let a_j, b_j, c_j be positive integers for $1 \leq j \leq N$. Assume that for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, where $1 \leq j \leq N$.

(8) A checkerboard has 4 rows and 7 columns. Choosing two or more successive rows and two or more successive columns and taking only the squares in those rows and columns gives a subboard. Suppose that each of the 21 squares is colored either black or white. Show that there is a subboard all of whose corners are the same color.

(9) A group of n people play a round-robin arm-wrestling tournament. In other words, every pair of people arm-wrestle exactly once. Each match ends in either a win or a loss. Show that it is possible to label the players $P_1, P_2, P_3, ..., P_n$ in such a way that P_1 defeated P_2, P_2 defeated $P_3, \ldots, P_{n-1}$ defeated P_n.