MAT 495 WEEK 1: MORE MATHEMATICAL INDUCTION QUESTIONS

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These are way too many problems to consider. Just pick a few problems you like and play around with them. You are not allowed to try a problem that you already know how to solve. Otherwise, work on the problems you want to work on.

1. The Hints:

Work in groups. Try small cases. **Plug in small numbers.** Do examples. Look for patterns. Draw pictures. Use LOTS of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

2. Mathematical Induction

Suppose that $P(n)$ is a statement for each positive integer n. The principle of mathematical induction states that if

- $P(1)$ is true - this is usually called the *base case*.
- For each positive integer n, if $P(n)$ is true, then $P(n+1)$ is true as well,

then $P(n)$ is true for all positive integers n.

2.1. More Induction Problems.

(1) Prove that $4^n - 1$ is divisible by 3 for every positive integer n.

(2) Write number $1, 1$ on the blackboard. Then, every minute, write the sum of 2 numbers between them. So after 1 minute you get $1, 2, 1$, after 2 minutes $1, 3, 2, 3, 1$, after 3 minutes $1, 4, 3, 5, 2, 5, 3, 4, 1$. After 100 minutes have passed, what is the sum of all the numbers on the blackboard?

(3) Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Prove that every positive integer k can be written as $k = \sum_{i=1}^{n} a_i \cdot i!$ for integers a_i satisfying $0 \leq a_i \leq i$.

(4) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

(5) $2n$ points are drawn on a circle, and $n^2 + 1$ pairs of points are joined by an edge. Prove that at least 1 triangle is formed.

(6) Given $T_1 = 2$, and $T_{n+1} = T_n^2 - T_n + 1$ for $n > 0$, so the sequence T_n starts $2, 3, 7, 43, \ldots$. Prove that any two distinct terms in this sequence are coprime. That is, if $n > m$, then T_n and T_m are coprime.

(7) A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking either
   (a) one bean from a heap, provided at least two beans are left behind in that heap, or
   (b) a complete heap of two or three beans.

   The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy. (Nathan Pflueger, 1995B5)

2.2. **Problems from Last Time.**

(1) Show that $1 + 2 + \cdots + n = n(n+1)/2$.
(2) Show that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$.
(3) Find a formula for the sum of the first n odd numbers.
(4) Prove that for $n \geq 12$, then n cents of postage can be made up using only 7 cent and 3 cent stamps.
(5) Prove that for $n \geq 6$ a square can be dissected into n squares, not necessarily all of the same size.
(6) Let r be a real number such that $r + \frac{1}{r}$ is an integer. Prove that $r^{100} + \frac{1}{r^{100}}$ is also an integer.
(7) Prove the (very useful) principle of **strong induction**, which goes as follows:

    Suppose that $P(n)$ is a statement for each positive integer n. The principle of mathematical induction states that if
    - $P(1)$ is true - this is usually called the *base case*.
    - For each positive integer n, if $P(1), P(2), \ldots, P(n)$ are all true, then $P(n+1)$ is true as well,

    then $P(n)$ is true for all positive integers n.
(8) The Fiobonacci sequene F_n is defined as follows. $F_1 = 1, F_2 = 1$ and for $n > 0$, $F_{n+2} = F_{n+1} + F_n$. So the sequences starts $1, 1, 2, 3, 5, 8, 13$. Prove that for any positive integer n, $F_n^2 + F_{n+1}^2 = F_{2n+1}$.
(9) For any set T whose elements are positive integers, define $f(T)$ to be the square of the product of the elements of T. For example, if $T = \{1, 2, 5\}$ then $f(T) = (1 \cdot 2 \cdot 5)^2 = 100$. For any positive integer n, consider all non-empty subsets S of $\{1, 2, \ldots, n\}$ that do not contain two consecutive positive integers. Prove that if one sums $f(S)$ over all of these subsets, one gets $(n+1)! - 1$.
(10) Prove the arithmetic mean - geometric mean inequality (AM-GM): Suppose $a_1, \ldots a_n$ are n positive real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{\frac{1}{n}}.$$

(a) First do the case of $n = 2$.
(b) Then do the case where n is a power of 2.
(c) Prove that if the statement is true for n then it is true for $n-1$.
(d) Prove that if the statement is true for n then it is true for $2n$.
(e) Conclude!

(11) We call a property *wacky* of the positive integers as follows: 1 and 2 are wacky by definition. For any positive integer n thats at least 3, we say that n is wacky if there is a unique way to write n as a sum of two distinct wacky numbers smaller thatn n. So 3 is wacky, 4 is wacky, 5 is not wacky, since 5=1+4=2+3, 6 is wacky, etc...

Prove that there are infinitely many wacky numbers.

(12) given a sequence $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$. Prove that for any positive integer k, the sequence $a_1, a_2, \ldots$ becomes constant modulo k. *for example, the sequence starts* $1, 2, 4, 16, 65536$ *which modulo* 3 *is* $1, 2, 1, 1, 1$, *and it continues being* 1 *modulo* 3.

(13) Every road in the country Graphsville is one way, and every pair of cities is connected by exactly one road. Show that there is a city which can be reached by every other city either directly or by going through at most one other city.