MAT 495 HOMEWORK 6: PROBABILITY AND GRAPH THEORY

JACOB TSIMERMAN

Write up and submit any 4 problems *with proofs* by Friday, December 11th. Points will be deducted for missing cases, gaps in the proof, algebra mistakes, and errors in reasoning, so please write very carefully!

If you can't solve 4 problems, you can submit partial work, but please do not submit work on more than 4 problems.

1. Problems

(1) You flip a fair coin 7 times. What is the probability that you get at least one head?

(2) Two teams are playing a best-of-7 in the world series, in which the first team to win 4 matches wins. Every individual game they play is equally likely to be won by either team. Is the game more likely to end in 6 matches or in 7?

(3) An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$. Your objective is to select n in an odd number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that your chance of winning is greater than $\frac{2}{3}$.

(4) A fair coin is tossed 1000 times. What is the probability that at some points two heads will be turn up in succession?

(5) Let G be a graph with 9 vertices. The degree of each vertex is either 5 or 6. Prove that there are either 5 vertices of degree 6 or 6 vertices of degree 5.

(6) Let G be a graph with n vertices, such that each vertex has degree at least $\frac{n-1}{2}$. Prove that if u and v are vertices in G with no edge between them, then there is a third vertex w such that uw and vw are both edges in G.

(7) Let G be a graph with n vertices such that each vertex has degree at least $\frac{n-1}{2}$. Prove that one can order the vertices $v_1, v_2, \ldots, v_n$ such that $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ are all edges (in other words, G has a *hamiltonian path*).

(8) Let n be a positive integer. A *partition* of n is a sequence of positive integers $(a_1, \ldots, a_k)$ such that $a_1 \geq a_2 \cdots \geq a_k$ such that $a_1 + \cdots + a_k = n$. The partition is called *odd* if all the a_i are odd. The partition is called *distinct* if $a_1 > a_2 > \cdots > a_k$.

Prove that for everyn n, the number of odd partitions is equal to the number of distinct partitions.