MAT 495 HOMEWORK 4: GAME THEORY

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Write up and submit any 4 problems *with proofs* by November 6th. Homework should be submitted to Asif Zaman's Mailbox, in the Bahen center, Math office, or in class. Points will be deducted for missing cases, gaps in the proof, algebra mistakes, and errors in reasoning, so please write very carefully!

If you can't solve 4 problems, you can submit partial work, but please do not submit work on more than 4 problems.

1. Problems

(1) Suppose f_n is a sequence with $f_0 = 1, f_1 = 1$ and satisfying the recurrence relation $f_{n+2} = 3f_{n+1} - 2f_n$. Find f_{100}.

(2) Suppose f_n is a sequence with $f_0 = 2$ and satisfyig the recurrence relation $f_n - f_{n-1}^2 - 2f_{n-1} = 0$. Find f_{100}.

(3) Jacob is going on vacation for 9 days. Every day he will surf, do math, or rest. On any given day he does just one of these three things. He never does surfing and math on consecutive days. How many schedules are possible for the holiday?

(4) There are n lines in the plane, no two of them parallel. These lines divide the plane into $B(n)$ regions (some infinite, some may be finite). For example, $B(1) = 2, B(2) = 4, B(3) = 7$. Find $B(2015)$.

(5) An L-piece, consisting of 3 1×1 squares, consists of a 2×2 square with one of the coroners removed. L-pieces may be rotated, so they have 4 possible orientations. Find the number of ways to tile a 3×2000 rectangle by L-pieces.

(6) Let F_n be the Fiobonacci sequence, so that $F_1 = F_2 = 1$, and $F_n + F_{n+1} = F_{n+2}$. Prove that $\sum_{n=1}^{2015} \frac{F_n}{2^n} < 2$.

(7) f_n is a sequence such that $f_1 = 1$, and for every positive integer n,

$$f_1 + f_2 + \cdots f_n = n^2 f(n).$$

Determine $f(2015)$.

(8) (Hard!) A sequence f_n is *eventually periodic* if there exists positive integers m, N such that $f_{n+m} = f_n$ for all $n > N$. Intuitively, a sequence is eventually periodic if it starts repeating at some point. Suppose that f_n is a sequence of 0's and 1's such that that are no more than 2015 distinct blocks of 2015 consecutive terms. Prove that f_n is eventually periodic (Hint: use induction!)

(9) (Hard!) The positive integers $b_1, b_2, \dots, b_n, \dots$ are all less than 2015, and for all m, n its true that $b_m + b_n$ is divisible by b_{m+n}. Prove that the sequence b_n is eventually periodic.