MAT 495 HOMEWORK 1: INDUCTION PROBLEMS

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Write up and submit any 4 problems *with proofs* by Friday, October 9th. Points will be deducted for missing cases, gaps in the proof, algebra mistakes, and errors in reasoning, so please write very carefully!

If you can't solve 4 problems, you can submit partial work, but please do not submit work on more than 4 problems.

1. Problems

(1) Write number $1, 1$ on the blackboard. Then, every minute, write the sum of 2 numbers between them. So after 1 minute you get $1, 2, 1$, after 2 minutes $1, 3, 2, 3, 1$, after 3 minutes $1, 4, 3, 5, 2, 5, 3, 4, 1$. After 100 minutes have passed, what is the sum of all the numbers on the blackboard?

(2) Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$. Prove that every positive integer k can be written as $k = \sum_{i=1}^{n} a_i \cdot i!$ for integers a_i satisfying $0 \leq a_i \leq i$.

(3) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

(4) $2n$ points are drawn on a circle, and $n^2 + 1$ pairs of points are joined by an edge. Prove that at least 1 triangle is formed.

(5) Show that $1 + 2 + \cdots + n = n(n+1)/2$.

(6) Show that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(7) Prove that for $n \geq 6$ a square can be dissected into n squares, not necessarily all of the same size.

(8) For any set T whose elements are positive integers, define $f(T)$ to be the square of the product of the elements of T. For example, if $T = \{1, 2, 5\}$ then $f(T) = (1 \cdot 2 \cdot 5)^2 = 100$. For any positive integer n, consider all non-empty subsets S of $\{1, 2, \ldots, n\}$ that do not contain two consecutive positive integers. Prove that if one sums $f(S)$ over all of these subsets, one gets $(n+1)! - 1$.

(9) We call a property *wacky* of the positive integers as follows: 1 and 2 are wacky by definition. For any positive integer n thats at least 3, we say that n is wacky if there is a unique way to write n as a sum of two distinct wacky numbers smaller thatn n. So 3 is wacky, 4 is wacky, 5 is not wacky, since 5=1+4=2+3, 6 is wacky, etc...

Prove that there are infinitely many wacky numbers.

(10) given a sequence $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2^{a_n}$. Prove that for any positive integer k, the sequence $a_1, a_2, \ldots$ becomes constant modulo k.

for example, the sequence starts $1, 2, 4, 16, 65536$ *which modulo* 3 *is* $1, 2, 1, 1, 1$, *and it continues being* 1 *modulo* 3.

(11) Every road in the country Graphsville is one way, and every pair of cities is connected by exactly one road. Show that there is a city which can be reached by every other city either directly or by going through at most one other city.

(12) (Hard!) Let n be a positive integer. You are given $\frac{n(n+1)}{2}$ stones, divided into piles of various sizes. Each minute, you take one stone from each existing pile, and group them together into a new pile. Prove that after enough time, you will have exactly one pile of size i for each integer i between 1 and n.