MAT 495 FINAL EXAM

JACOB TSIMERMAN

This exam is due at 5 PM Monday, December 14, 2015. Please hand in the exam to the main office at the math department, located at room 6290, Bahen center, 40 St.George Street.

Each question will be worth 10 points, for a maximum of 100 points, but 80 points will be enough for a perfect score. You may not discuss the problems with any other student, nor use any online resources. If you have clarification questions (that is, you do not understand the wording of a problem) please e-mail the professor.

(1) There are 10 points inside an equilateral triangle of side length 3. Prove that two of the points are within distance at most 1 of each other.

(2) On a table there are 2015 tokens. Alice and Bob take turns removing tokens, with Alice moving first. On each turn, $7, 8, 9, 10, 11$ or 12 tokens may be removed. The game ends when no more moves are possible. If a player manages to take the last token on their move, they are the winner. Else, the game is a draw. Prove that if Alice and Bob play perfectly, the game will be a draw.

(3) Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there exists an integer n greater than 1000000 such that $P(n)$ is not a prime number.

(4) In a 100×100 square grid there is one number written into each 1×1 cell, such that each of the numbers $1, 2, \ldots, 100$ is written exactly 100 times. Prove that there is either a row or a column with at least 10 distinct numbers written in its cells.

(5) Let b_n be the last digit of the number $\sum_{a=1}^{n} a^a = 1^1+2^2+3^3+\cdots+n^n$. Prove that $b_{n+100} = b_n$.

(6) Let n be a positive integer. Prove that
$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

(7) There are n positive integers written on the blackboard. A move consists of choosing two positive integers a, b from the blackboard such that neither divides the other, and replace them with their greatest common divisor and their lowest common multiple, $\gcd(a, b)$ and $\text{lcm}(a, b)$. For example, you can replace $(5, 7)$ with $(1, 35)$. Every minute, if you can, you perform a move. Prove that eventually, you can no longer perform any moves.

(8) Let $P(x) = \prod_{i=1}^{2015}(x-i)^2 + 1$. Prove that P cannot be written as a product of two other non-constant polynomials with integer coefficients.

(9) Let G be a connected graph with k edges (every edge connects two distinct vertices of G). For each vertex v, let $S(v)$ be the labels of all the edges that have v as an endpoint. Prove that you can label the edges $1, 2, \ldots, k$ such that for every vertex v for which $|S(v)| \geq 2$, the greatest common divisor of all the elements of $S(v)$ is 1.

(10) 2015 pennies are distributed among 3 boxes. If two of the boxes A, B have p, q pennies respectively, such that $p \geq q$, you may take q pennies from box A and transfer them to box B. Prove that after a finite number of such moves, you may ensure that only two boxes have pennies in them.