MAT382: HOMEWORK SET #3

DUE: MONDAY FEBRUARY 26, 2018

You are expected to write complete proofs to the following problems - leaps of logic or unproven assumptions will be penalized. You may use any statement we've proven in class freely. Please ask me if any question is unclear. Every question is worth 10 points, making the homework out of 50.

(1) Let $a, k \geq 2$ be positive integers. Prove that k divides $\phi(a^k - 1)$.
(2) Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial with integer coefficients in n variables, and p be a prime number. Suppose that $f(a_1, a_2, \ldots, a_n) = 0$ and there exists at least one $j \in \{1, 2, \ldots, n\}$ with $\frac{\partial f}{\partial x_j}(a_1, \ldots, a_n) \neq 0 \mod p$. Prove that the congruence $f(x) \equiv 0 \mod p^k$ has at least one integer solution for every positive integer k.
(3) Suppose p, q are primes with $p = 2q + 1$, and m is a positive integer with $m < \sqrt{p+1}$ such that $\left(\frac{m}{p}\right) = -1$. Prove that m is a primitive root mod p.
(4) Let p be a prime number. Prove that for every element $t \in \mathbb{Z}/p\mathbb{Z}$ there exist $x, y, z \in \mathbb{Z}/p\mathbb{Z}$ with $x^3 + y^3 + z^3 = t$.
(5) Let p be an odd prime and a a positive integer co-prime to p. Extend the Jacobi symbol by defining $\left(\frac{m}{n}\right)$ to be 0 if $\gcd(m, n) > 1$. Prove that
$$\sum_{n=1}^{p} \left(\frac{n^2 + a}{p}\right) = -1.$$
(6) Let $F(n)$ be the number of triples of positive integers (a, b, c) such that $\gcd(a, b, c) = 1$ and $\max(a, b, c) \leq n$. Compute $\lim_{n\to\infty} \frac{F(n)}{n}$.
(7) (Bonus Question - 1% on the course) Prove that there do not exist positive integers r, s, t such that
$$\frac{r^2 + s^2 + t^2}{3(rs + st + rt)}$$
is an integer.