

ON THE IRREDUCIBILITY OF p -ADIC BANACH PRINCIPAL SERIES OF p -ADIC REDUCTIVE GROUPS

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ABSTRACT. Suppose that G is the group of F -points of a connected reductive group over F , where F/\mathbb{Q}_p is a finite extension. We study the (topological) irreducibility of principal series of G on p -adic Banach spaces. For unitary inducing representations we obtain an optimal irreducibility criterion, and for $G = \mathrm{GL}_n(F)$ (as well as for arbitrary split groups under slightly stronger conditions) we obtain a variant of Schneider’s conjecture [Sch06, Conjecture 2.5]. In general we reduce the irreducibility problem to smooth inducing representations and almost simple simply-connected G . Our methods include locally analytic representation theory, the bifunctor of Orlik–Strauch, translation functors, as well as new results on reducibility points of smooth parabolic inductions.

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The first-named author was supported by JSPS KAKENHI Grant Number 18H01107.
The second-named author was partially supported by an NSERC grant.

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1. INTRODUCTION

Suppose that F/\mathbb{Q}_p is a finite extension and $G = \underline{G}(F)$ the group of F -points of a connected reductive group \underline{G} over F . This paper concerns the continuous representations of G on p -adic Banach spaces over a coefficient field C that is a (sufficiently large) finite extension of \mathbb{Q}_p . Such Banach representations were introduced in the work of Schneider–Teitelbaum [ST02] and play a fundamental role in the p -adic Langlands program (see for example [Bre04], [BS07], [Col10], [Eme11], [Paš13], [CEG⁺16]), particularly the unitary such representations, i.e. those that admit a G -invariant defining norm. Little has been known about Banach representations outside the group $\mathrm{GL}_2(\mathbb{Q}_p)$ so far. The main goal of this paper is to greatly extend our knowledge about the (topological) irreducibility of Banach principal series representations. In particular, we prove a variant of Schneider’s conjecture [Sch06, Conjecture 2.5].

Fix a minimal parabolic subgroup $\underline{B} = \underline{Z}\underline{U}$ with Levi subgroup \underline{Z} and unipotent radical \underline{U} . Let $\underline{P} = \underline{L}\underline{N}$ be a parabolic subgroup containing \underline{B} with Levi subgroup \underline{L} containing \underline{Z} and unipotent radical \underline{N} . We write $B := \underline{B}(F)$, etc. If σ is a continuous representation of L on a finite-dimensional C -vector space, then we inflate σ to P and form the parabolic induction

$$(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}} := \{f: G \rightarrow \sigma \text{ cts.} : f(p'g) = \sigma(p')f(g) \text{ for any } g \in G, p' \in P\},$$

which carries a natural Banach topology making it into an (admissible) Banach representation of G under right translation that we call a *Banach principal series*. We show moreover in Proposition 2.50 that it is topologically of finite length, confirming an expectation of [Sch06, §2]. (For technical reasons we will assume for the remainder of the introduction that $p > 2$, resp. $p > 3$, if the absolute root system of \underline{G} has irreducible components of type B , C or F_4 , resp. G_2 .) From now on we will assume that σ is absolutely irreducible.

1.1. Unitary case. Our first theorem gives an irreducibility criterion for general \underline{G} in the case where σ is unitary. Let \underline{S} be the maximal split subtorus of the center of \underline{Z} . Let Δ (resp. Δ_L) be the set of simple roots of \underline{S} in \underline{B} (resp. in $\underline{B} \cap \underline{L}$). For any $\alpha \in \Delta$ let \underline{L}_α denote the Levi subgroup containing \underline{Z} with $\Delta_{\underline{L}_\alpha} = \{\alpha\}$. Moreover, let \underline{L}'_α denote the smallest normal subgroup of \underline{L}_α generated by $\underline{U} \cap \underline{L}_\alpha$. (This is not an algebraic subgroup in general.)

Theorem 1.1 (Theorem 3.33). *Let σ be a finite-dimensional absolutely irreducible unitary Banach representation of L . Then the following are equivalent:*

- (i) $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is reducible;
- (ii) there exists $\alpha \in \Delta \setminus \Delta_L$ such that $\sigma|_{Z \cap \underline{L}'_\alpha}$ is trivial;
- (iii) the representation σ extends to a continuous representation of a Levi subgroup strictly containing L .

We briefly explain the easy implications (ii) \Rightarrow (iii) \Rightarrow (i). If (ii) holds, let $\underline{L}_1 \supset \underline{Z}$ be the Levi subgroup with $\Delta_{\underline{L}_1} = \Delta_L \sqcup \{\alpha\}$. If σ is trivial on $Z \cap \underline{L}'_\alpha$, then σ has a unique (continuous) extension to \underline{L}_1 which is trivial on $N \cap \underline{L}_1$ [AHHV17,

II.7 Proposition], and we get (iii). If (iii) holds, σ has a continuous extension to a larger Levi $L_1 \supsetneq L$, which we still denote by σ . Let \underline{P}_1 be the parabolic subgroup containing \underline{P} with Levi subgroup \underline{L}_1 . Then $(\text{Ind}_{\underline{P}_1}^G \sigma)^{\text{cts}} \hookrightarrow (\text{Ind}_{\underline{P}}^G \sigma)^{\text{cts}}$ is a proper closed subrepresentation and so (i) holds.

In fact, we obtain the equivalence in Theorem 1.1 under the weaker assumption that σ lies in a certain closed cone, defined as follows. Let $\omega_{\sigma_{U \cap L}} : S \rightarrow C^\times$ be the central character of the coinvariant representation $\sigma_{U \cap L}$ (which is absolutely irreducible). Then $|\omega_{\sigma_{U \cap L}}|_C : S \rightarrow \mathbb{R}_{>0}^\times$ is an unramified character of S (where $|\cdot|_C$ denotes the absolute value of C with $|p|_C = p^{-1}$) and we let $e(\omega_{\sigma_{U \cap L}}) \in X^*(\underline{S}) \otimes \mathbb{R}$ denote its Harish-Chandra parameter. Then it suffices for our proof to demand that $e(\omega_{\sigma_{U \cap L}})$ be dominant (or even that it is almost dominant in a precise and explicit sense, see Remark 3.67). We also have a version of this theorem where we only demand a certain dominance condition on the central character ω_σ of σ .

1.2. Schneider's conjecture. Dropping the assumption that σ is unitary, we prove a variant of Schneider's conjecture for the group $\text{GL}_n(F)$ and, under slightly weaker conditions, for general split (or even quasisplit) groups G .

To state our results, let δ_B denote the modulus character of B . We say that a character $\lambda : F^\times \rightarrow C^\times$ is *non-positive algebraic* if it is of the form $\lambda(x) = \prod_{\kappa : F \rightarrow C} \kappa(t)^{a_\kappa}$ for some $(a_\kappa) \in \mathbb{Z}_{\leq 0}^{\text{Hom}(F, C)}$. For $1 \leq i < j \leq n$ let $\alpha_{i,j}^\vee : \mathbb{G}_m \rightarrow \text{SL}_n$ denote the coroot sending t to $\text{diag}(1, \dots, t, \dots, t^{-1}, \dots, 1)$ with t (resp. t^{-1}) appearing in the i -th (resp. j -th) entry. Let $|\cdot|_F$ denote the normalized absolute value of F .

Theorem 1.2 (Theorem 3.9). *Let $G = \text{GL}_n(F)$ or $\text{SL}_n(F)$, B the upper-triangular Borel subgroup, and Z the diagonal maximal torus. Let $\chi : Z \rightarrow C^\times$ be a continuous character. If $(\text{Ind}_B^G \chi)^{\text{cts}}$ is reducible, then there exist $1 \leq i < j \leq n$ and non-positive algebraic characters λ_k for $i \leq k < j$ such that*

- $\chi \delta_B^{-1/2} \circ \alpha_{k,k+1}^\vee \equiv \lambda_k$ near the identity for any $i \leq k < j$, and
- $\chi \delta_B^{-1/2} \circ \alpha_{i,j}^\vee = |\cdot|_F^{-1} \prod_{k=i}^{j-1} \lambda_k$.

In particular, there exists a positive root $\alpha_{i,j}$ and a non-positive algebraic character λ such that $\chi \delta_B^{-1/2} \circ \alpha_{i,j}^\vee = |\cdot|_F^{-1} \lambda$. As in Theorem 1.1(ii) \Rightarrow (i), if the condition of the previous sentence holds with $j = i + 1$, then $(\text{Ind}_B^G \chi)^{\text{cts}}$ is reducible. When $n = 3$ we show that these are in fact the only possible reducibilities, see our companion paper [AH].

We have a slightly weaker theorem valid for any split connected reductive group (adding a Weyl group regularity condition).

Theorem 1.3 (Theorem 3.10). *Suppose that \underline{G} is split. Let $\chi : Z \rightarrow C^\times$ be a continuous character. Let $\underline{P} = \underline{LN}$ be the largest standard parabolic such that for all roots $\alpha : \underline{Z} \rightarrow \mathbb{G}_m$ occurring in \underline{L} we have $\chi \delta_B^{-1/2} \circ \alpha^\vee \equiv \lambda_\alpha$ near the identity, for some non-positive algebraic character λ_α . Assume that for all $w \in N_L(Z) \setminus Z$ there exists a root α of \underline{L} such that*

- $(\chi \delta_B^{-1/2} \circ w^{-1} \alpha^\vee) \cdot (\chi \delta_B^{-1/2} \circ \alpha^\vee)^{-1} \neq \lambda_{w^{-1} \alpha} \lambda_\alpha^{-1}$.

If $(\text{Ind}_B^G \chi)^{\text{cts}}$ is reducible, then there exists a positive root α occurring in \underline{L} such that

- $\chi \delta_B^{-1/2} \circ \alpha^\vee = |\cdot|_F^{-1} \lambda_\alpha$.

To compare Theorems 1.2, 1.3 with [Sch06, Conjecture 2.5], let us assume that in addition that \underline{G} is semisimple simply connected (as is assumed there) and in addition for simplicity that $F = \mathbb{Q}_p$. Let $\delta^{1/2} : \underline{Z} \rightarrow \mathbb{G}_m$ denote the half-sum of positive roots, which is integral by our assumption on \underline{G} , and note that $|\delta^{1/2}|_F =$

$\delta_B^{1/2}$. Then [Sch06, Conjecture 2.5] is equivalent to: if $(\text{Ind}_B^G \chi)^{\text{cts}}$ is absolutely reducible, then there exists a positive root α of \underline{G} such that

- $\chi \delta^{-1/2} \circ \alpha^\vee = (\cdot)^{-1} \lambda_\alpha$ for some non-positive algebraic character λ_α .

(If α is a simple root, this is equivalent to $\chi \delta_B^{-1/2} \circ \alpha^\vee = |\cdot|_F^{-1} \lambda_\alpha$, exactly as in Theorem 1.3.) We remark that our conditions/conclusion in Theorems 1.2, 1.3 are compatible with the equivalence in Theorem 1.4 below.

We obtain more refined versions of Theorem 1.3 for any classical quasisplit group in §3.3. (These results are stated for smooth σ , for simplicity.)

For general reductive \underline{G} one may wonder whether $(\text{Ind}_B^G \sigma)^{\text{cts}}$ is absolutely reducible if and only if $(\text{Ind}_{B \cap L_\alpha}^{L_\alpha} \sigma)^{\text{cts}}$ is reducible for some simple root α . Our evidence is rather limited, consisting mostly of Theorem 1.1 and our result for $\text{GL}_3(F)$ [AH]. The group L_α is of semisimple rank 1, and we obtain an optimal irreducibility criterion for $(\text{Ind}_{B \cap L_\alpha}^{L_\alpha} \sigma)^{\text{cts}}$, at least when σ is simple as \mathfrak{z}_C -module (which holds when $\dim_C \sigma = 1$) or when the unipotent radical \underline{U} is abelian. See Theorem 3.68 and also Corollary 3.70.

1.3. A criterion and some reductions. By replacing \underline{G} by $\text{Res}_{F/\mathbb{Q}_p} \underline{G}$ we may and will assume that $F = \mathbb{Q}_p$. Then the action of L on σ becomes locally analytic and we can study the Banach representation $(\text{Ind}_P^G \sigma)^{\text{cts}}$ by means of its dense subspace of locally analytic vectors

$$(\text{Ind}_P^G \sigma)^{\text{an}} := \{f: G \rightarrow \sigma \text{ loc. an.} : f(p'g) = \sigma(p')f(g) \text{ for any } g \in G, p' \in P\},$$

see [ST02], [Eme17]. Note that $(\text{Ind}_P^G \sigma)^{\text{an}}$ naturally carries a compact type topology and becomes an admissible locally analytic representation of G in the sense of Schneider–Teitelbaum [ST03]. If σ is smooth and we replace locally analytic functions by locally constant functions, then we obtain a smooth subrepresentation $(\text{Ind}_P^G \sigma)^{\text{sm}}$ of $(\text{Ind}_P^G \sigma)^{\text{an}}$.

Orlik and Strauch [OS15], [OS14b] introduced a beautiful theory to understand the structure of the locally analytic principal series $(\text{Ind}_P^G \sigma)^{\text{an}}$. (In their papers they restrict to split groups \underline{G} . See Appendix A for the general case, and note that another treatment will appear in the revised version of [OS14b].) We just recall some basic definitions for now. Let \mathfrak{g} denote the Lie algebra of G , $\mathfrak{g}_C := \mathfrak{g} \otimes_F C$, and likewise for other subgroups of G . For any parabolic subgroup \underline{P} , Orlik–Strauch define artinian abelian categories $\mathcal{O}^{\mathfrak{p}}$, an adaptation of a parabolic BGG category \mathcal{O} over C whose objects consist of certain finitely generated $U(\mathfrak{g}_C)$ -modules, and \mathcal{O}^P whose objects consist of pairs (M, τ) with $M \in \mathcal{O}^{\mathfrak{p}}$ and τ a locally finite locally analytic action of P on M satisfying $p' \circ X \circ p'^{-1} = \text{Ad}(p')(X)$ for $p' \in P$, $X \in \mathfrak{g}_C$ which lifts the given action of \mathfrak{p}_C (see §2.3 for the precise definition). If $\underline{Q} \supset \underline{P}$ is another parabolic subgroup, then $\mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$ and $\mathcal{O}^Q \subset \mathcal{O}^P$ are naturally full subcategories, and we say that $M \in \mathcal{O}^P$ is *equimaximal* if for any parabolic subgroup \underline{Q} containing \underline{P} we have $M \in \mathcal{O}^Q$ if and only if $M|_{\mathfrak{g}_C} \in \mathcal{O}^{\mathfrak{q}}$. For any simple object $W \in \mathcal{O}^L$ there is a unique simple object $\underline{L}(W) \in \mathcal{O}^P$ such that $\underline{L}(W)^N \cong W$ (Lemma 2.11). (Here, \mathcal{O}^L is the category \mathcal{O}^G with $G = L$. Its objects are automatically finite-dimensional.)

Recall that σ is an absolutely irreducible finite-dimensional continuous representation of L . Then σ lies in \mathcal{O}^L (recall that $F = \mathbb{Q}_p$ now), as σ is automatically locally \mathbb{Q}_p -analytic. Assume now that $\sigma \cong \sigma_0 \otimes \tau$ for some $\sigma_0 \in \mathcal{O}^L$ whose underlying \mathfrak{l}_C -module is simple and some smooth L -representation τ such that moreover $\underline{L}(\sigma'_0)$ is equimaximal, where σ'_0 denotes the dual of σ_0 . Such a decomposition of σ always exists when the derived subgroup G^{der} is simply connected by Lemma 2.51 and is unique up to smooth characters of L_Q in general, where $\underline{Q} = \underline{L}_Q \underline{N}_Q$ is the largest parabolic subgroup containing \underline{P} such that $\underline{L}(\sigma'_0) \in \mathcal{O}^Q$.

We then have the following theorem, which is fundamental to our work.

Theorem 1.4 (Theorem 2.56, Corollary 2.54). *The following are equivalent:*

- (i) $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible;
- (ii) $(\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}$ is irreducible;
- (iii) $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$ is irreducible;
- (iv) any irreducible subrepresentation of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$.

In particular, $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible if $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is irreducible, by part (iv).

Based on this equivalence we would expect that it is extremely difficult to find a precise irreducibility criterion of Banach principal series, even for $G = \text{GL}_n(F)$. The problem is that the submodule structure of the smooth principal series $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is badly understood in general, and even when it is understood it is challenging to prove the p -adic density of a proper smooth subrepresentation.

Corollary 1.5 (Corollary 2.57). *Each of the following conditions implies the next:*

- (i) $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} \sigma'_0$ is irreducible as $U(\mathfrak{g}_C)$ -module;
- (ii) $Q = P$;
- (iii) $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible.

By Theorem 1.4 we can reduce the irreducibility problem of $(\text{Ind}_P^G \sigma)^{\text{cts}}$ to the case where σ is smooth. The following general result on intertwiners and some Clifford theory furthermore allows us to reduce to the case where \underline{G} is almost simple and simply connected (and isotropic), see Propositions 2.42 and 2.59.

Proposition 1.6 (Proposition 2.37). *Suppose that $P = \underline{L}N$ is a parabolic subgroup and σ, τ are Banach representations of L . Then the natural map*

$$\text{Hom}_L^{\text{cts}}(\sigma, \tau) \rightarrow \text{Hom}_G^{\text{cts}}((\text{Ind}_P^G \sigma)^{\text{cts}}, (\text{Ind}_P^G \tau)^{\text{cts}})$$

is an isomorphism.

1.4. Proof of Theorem 1.1. For simplicity we will assume that $P = B$. We first make a reduction to $F = \mathbb{Q}_p$ and $\underline{G}^{\text{der}}$ simply-connected, so in particular we have a decomposition $\sigma = \sigma_0 \otimes \tau$ as in §1.3. Then Theorem 1.4 implies that if $(\text{Ind}_B^G \sigma)^{\text{cts}}$ is reducible, then $(\text{Ind}_{B \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is reducible, where Q is defined as in §1.3. By relabeling we may assume that $Q = G$, i.e. $\underline{L}(\sigma'_0) \in \mathcal{O}^G$. This implies that, up to twisting σ by a locally analytic character of G , we may assume that σ_0 is algebraic with antidominant central character. Write $(\text{Ind}_B^G \tau)^{\text{sm}}$ as normalized induction $(\text{n-Ind}_B^G \tau \delta_B^{-1/2})^{\text{sm}}$. Using that σ is unitary we deduce that $\tau \delta_B^{-1/2}$ is Weyl group regular. We now fix an isomorphism $\overline{\mathbb{C}} \cong \mathbb{C}$ and work over the complex numbers. Then a result of Harish-Chandra implies that there exists a reduced positive root α such that $(\text{n-Ind}_{B \cap L_\alpha}^{L_\alpha} \tau \delta_B^{-1/2})^{\text{sm}}$ is reducible, where \underline{L}_α is the Levi subgroup of semisimple rank 1 associated to α . We can write $\tau \delta_B^{-1/2} \cong \tau_u \delta_{B \cap L_\alpha}^s \eta$ with τ_u unitary (in the complex sense!), $s \in \mathbb{R}$, and η a positive-real unramified character of L_α . A result of Silberger shows that $-\frac{1}{2} \leq s \leq \frac{1}{2}$. There is a tension between τ being p -adic unitary and the Silberger bound that allows us to deduce that $s = -\frac{1}{2}$, α is simple, and σ_0 is trivial on $Z \cap L'_\alpha$. Crucially, the extreme bounds $s = \pm \frac{1}{2}$ imply that τ_u is trivial on $Z \cap L'_\alpha$. This was our guess, based on the available literature, but Jean-Loup Waldspurger kindly provided a beautiful argument in general, see Proposition 3.7. We then deduce that σ is trivial on $Z \cap L'_\alpha$.

However, the above sketch glosses over one very important point. In comparing real and p -adic absolute values it is essential to know that the real number s is in fact *rational*. For this we need to prove Corollary 1.8 below.

1.5. Harish-Chandra's μ -function and rationality of poles. The following results in smooth representation theory over \mathbb{C} may be of independent interest. Harish-Chandra's μ -function (or Plancherel measure) μ^G controls the reducibility points of smooth parabolic inductions, as recalled in subsection 3.1.

Suppose that \underline{G} is any connected reductive group over F , $\underline{P} = \underline{L}\underline{N}$ a maximal parabolic subgroup such that \underline{L} is an inner form of a group \underline{L}' satisfying

$$(\tilde{\underline{L}}')^{\text{der}} \subset \underline{L}' \subset \tilde{\underline{L}}',$$

where

$$\tilde{\underline{L}}' := \left(\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i} \right) / \underline{H}$$

for some finite extensions E_i/F , integers $n_i \geq 1$ and a central, induced subtorus \underline{H} .

Theorem 1.7 (Theorem 3.43). *Let σ be a discrete series representation of L . If $\mu^G(\sigma\delta_P^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$.*

Our proof uses global intertwining operators and the global Jacquet–Langlands correspondence [Bad08] to reduce to the case where \underline{G} is quasisplit, in which case the rationality follows from results of Shahidi [Sha90] (see Proposition 3.41), using that the supercuspidal support of σ is automatically generic. (This generalizes an argument of Muić–Savin [MS00], who used this method to compare μ -functions of G and its quasisplit inner form when \underline{G} is a hermitian quaternionic group of maximal Witt rank and \underline{P} is the Siegel parabolic. See also [Cho14] for further work in this direction.)

By the classification of almost simple rank one groups we obtain the following corollary.

Corollary 1.8 (Corollary 3.59). *Suppose that the adjoint group $\underline{G}^{\text{ad}}$ is almost simple of rank one over F . Let σ be a unitary supercuspidal representation of Z . If $\mu^G(\sigma\delta_B^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$.*

In fact, using [Sha90, Theorem 8.1] we can also bound the denominator and obtain an explicit finite list of possible poles s_0 , see Remark 3.61. We remark that there are 7 families of almost simple rank one groups, and Corollary 1.8 was only known in 4 cases previously, see Remark 3.60.

We also obtain the following corollary from our argument. Here, $\tilde{\underline{L}}$ is an inner form of \underline{L}' , naturally obtained from the inner form \underline{L} of \underline{L}' , such that $\tilde{\underline{L}}^{\text{der}} \subset \underline{L} \subset \tilde{\underline{L}}$.

Corollary 1.9 (Corollary 3.44). *Suppose that σ_1, σ_2 are discrete series representations of L that are conjugate under the action of $\tilde{\underline{L}}$. Then $\mu^G(\sigma_1\delta_P^s) = \mu^G(\sigma_2\delta_P^s)$.*

This verifies [Cho14, Working Hypothesis 1.1] in our more general setup.

1.6. Proof of Theorems 1.2 and 1.3. Consider \underline{G} a split group over F and $\chi: Z \rightarrow C^\times$ a continuous character. Let us assume for simplicity that $\underline{G}^{\text{der}}$ is simply connected. By thinking of G as the \mathbb{Q}_p -points of $\text{Res}_{F/\mathbb{Q}_p} \underline{G}$ we can work over \mathbb{Q}_p when needed. As in §1.3 we write $\chi = \sigma_0\tau$ with $\underline{L}(\sigma_0) \in \mathcal{O}^Q$ (with Q maximal) and τ smooth. It is not hard to see that $Q = P$ in the notation of Theorem 1.3. We now prove the contrapositive of Theorems 1.2 and 1.3. By Theorem 1.4 we may reduce to the case where $Q = G$, and it suffices to show that any irreducible subrepresentation of $(\text{Ind}_B^G \tau)^{\text{sm}}$ is dense in $(\text{Ind}_B^G \tau)^{\text{cts}}$. As τ is smooth, the first bullet in Theorem 1.3 implies that $\tau\delta_B^{-1/2}$ is Weyl group regular and the second bullet in Theorem 1.3 becomes that $\tau\delta_B^{-1/2} \circ \alpha^\vee \neq |\cdot|_F^{-1}$ for all positive roots α . Work of Rodier [Rod81] then shows that $(\text{Ind}_B^G \tau)^{\text{sm}} = (\text{n-Ind}_B^G \tau\delta_B^{-1/2})^{\text{sm}}$ has an

irreducible socle that is moreover generic. (When $G = \mathrm{GL}_n(F)$ we do not need the regularity condition by Bernstein–Zelevinsky [BZ77].)

We then conclude by the following result.

Proposition 1.10. *Any generic subrepresentation of $(\mathrm{Ind}_B^G \tau)^{\mathrm{sm}}$ is dense in $(\mathrm{Ind}_B^G \tau)^{\mathrm{cts}}$.*

To prove the proposition, we first show by the geometric lemma (cf. Proposition 2.61) that any generic subrepresentation π of $(\mathrm{Ind}_B^G \tau)^{\mathrm{sm}}$ contains an element f that is supported on the big cell $B \backslash Bw_0U$, where w_0 is the longest Weyl group element. Then we deduce by a p -adic approximation argument (Lemma 2.2) that π is dense in $(\mathrm{Ind}_B^G \tau)^{\mathrm{cts}}$.

In fact, Proposition 1.10 generalizes to arbitrary G , where we say that a smooth representation π is *generic* if the twisted coinvariants $\pi_{U,\theta}$ are nonzero for some character $\theta: U \rightarrow C^\times$ that is non-trivial on each simple root subgroup. We then generalize Theorem 1.2 to $\mathrm{GL}_n(D)$, where D is a finite-dimensional division algebra, see Theorem 3.26. (Note that by Theorem 1.4 we may assume that the inducing representation σ is smooth.) For this we generalize Bernstein–Zelevinsky’s theory of derivatives [BZ77], cf. subsection 3.4.

1.7. Proof of Theorem 1.4. We recall that Orlik–Strauch [OS15], [OS14b] define a functor \mathcal{F}_P^G from $(\mathcal{O}^P)^{\mathrm{op}} \times \mathrm{Rep}^{\mathrm{adm}}(L)$ to locally analytic representations of G , where $\mathrm{Rep}^{\mathrm{adm}}(L)$ denotes the category of admissible smooth L -representations. This functor satisfies the following properties (see [OS14b] and subsection 2.3):

- The functor \mathcal{F}_P^G is exact in both arguments.
- Let $P' = L'N' \supset P$ be another parabolic subgroup. If $M \in \mathcal{O}^{P'}$ and $\pi \in \mathrm{Rep}^{\mathrm{adm}}(L)$, then $\mathcal{F}_P^G(M, \pi) \cong \mathcal{F}_{P'}^G(M, (\mathrm{Ind}_{P \cap L'}^{L'} \pi)^{\mathrm{sm}})$.
- Assume that $M \in \mathcal{O}^P$ is equimaximal with maximal parabolic P and $\pi \in \mathrm{Rep}^{\mathrm{adm}}(L)$. Assume that $M|_{\mathfrak{g}_C} \in \mathcal{O}^P$ is simple and π is irreducible. Then $\mathcal{F}_P^G(M, \pi)$ is irreducible.

For $W \in \mathcal{O}^L$ the generalized Verma module $\underline{M}(W)$ is defined to be $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W$, where P acts by $p'(X \otimes w) = \mathrm{Ad}(p')X \otimes p'w$ for $p' \in P$, $w \in W$ and $X \in \mathfrak{g}_C$. Then $\underline{M}(W) \in \mathcal{O}^P$, and if moreover $W \in \mathcal{O}^L$ is simple, then $\underline{M}(W)$ has $\underline{L}(W)$ as unique simple quotient. Then the following property of \mathcal{F}_P^G holds by construction:

- Suppose that $W \in \mathcal{O}^L$ and $\pi \in \mathrm{Rep}^{\mathrm{adm}}(L)$. Then $\mathcal{F}_P^G(\underline{M}(W), \pi) \cong (\mathrm{Ind}_P^G W' \otimes \pi)^{\mathrm{an}}$.

We now discuss the proof of Theorem 1.4. The implication (i) \Rightarrow (ii) is clear by exactness of parabolic induction, and (iii) \Rightarrow (iv) is obvious.

To explain why (iv) \Rightarrow (i), we assume for simplicity that $Q = G$ (the proof is a bit more involved in general). By the density of locally analytic vectors in Banach representations [ST02], it suffices to show that any irreducible closed subrepresentation of $(\mathrm{Ind}_P^G \sigma)^{\mathrm{an}}$ is dense in $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$. Note that $(\mathrm{Ind}_P^G \sigma)^{\mathrm{an}} \cong \mathcal{F}_P^G(\underline{M}(\sigma'_0), \tau)$. Let $V := \underline{L}(\sigma'_0)'$, which is by assumption in \mathcal{O}^G and in particular is a finite-dimensional locally analytic representation of G . The canonical P -linear surjection $V \rightarrow \sigma_0$ gives rise to a commutative diagram

$$\begin{array}{ccccccc} V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{sm}} & \hookrightarrow & V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{an}} & \xrightarrow{\sim} & (\mathrm{Ind}_P^G V \otimes \tau)^{\mathrm{an}} & \twoheadrightarrow & (\mathrm{Ind}_P^G \sigma)^{\mathrm{an}} \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{cts}} & \xrightarrow{\sim} & (\mathrm{Ind}_P^G V \otimes \tau)^{\mathrm{cts}} & \twoheadrightarrow & (\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}} \end{array}$$

It is not difficult to see (cf. Lemma 2.52) that the composition of the top row is injective with image $\mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau)$. Crucially, our generalization of a result of Breuil [Bre16] (based on [OS14a]) on locally analytic socles allows us to deduce that any

irreducible closed subrepresentation of $(\mathrm{Ind}_P^G \sigma)^{\mathrm{an}}$ is contained in $\mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau) \cong V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{sm}}$. It is then of the form $V \otimes \rho$ for some irreducible subrepresentation $\rho \subset (\mathrm{Ind}_P^G \tau)^{\mathrm{sm}}$. By (iv) we see that $V \otimes \rho$ is dense in $V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{cts}}$ and hence by the diagram $V \otimes \rho$ is dense in $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$, as desired.

It remains to explain (ii) \Rightarrow (iii). We relabel Q as G and let again $V := \underline{L}(\sigma'_0)' \in \mathcal{O}^G$. Let π be a nonzero closed subrepresentation of $(\mathrm{Ind}_P^G \tau)^{\mathrm{cts}}$. We consider a natural sequence

$$V \otimes \pi^{\mathrm{an}} \hookrightarrow V \otimes (\mathrm{Ind}_P^G \tau)^{\mathrm{an}} \xrightarrow{\sim} (\mathrm{Ind}_P^G V \otimes \tau)^{\mathrm{an}} \twoheadrightarrow (\mathrm{Ind}_P^G \sigma)^{\mathrm{an}}$$

of locally analytic representations and first use (ii) (comparing with a corresponding sequence of Banach representations) and locally analytic socles to show that the composition is surjective. Let $\chi : Z(\mathfrak{g}_C) \rightarrow C$ denote the infinitesimal character of $(\mathrm{Ind}_P^G \sigma)^{\mathrm{an}}$. Projecting onto generalized χ -eigenspaces we obtain a sequence

$$T(\pi^{\mathrm{an}}) \hookrightarrow T((\mathrm{Ind}_P^G \tau)^{\mathrm{an}}) \twoheadrightarrow (\mathrm{Ind}_P^G \sigma)^{\mathrm{an}},$$

whose composition is surjective, and where T is a suitable translation functor in the sense of [BG80] (it is an equivalence by [BG80]). By showing that the second map is an isomorphism (Proposition 2.48) and applying a quasi-inverse of T we obtain that $\pi^{\mathrm{an}} = (\mathrm{Ind}_P^G \tau)^{\mathrm{an}}$, which implies (iii), by density of locally analytic vectors.

1.8. Previous work. Theorems 1.1 and 1.2 were known for $\mathrm{GL}_2(\mathbb{Q}_p)$ by Schneider (and Teitelbaum) [Sch06, Proposition 2.6(i)], which was also based on locally analytic techniques. The infinitesimal irreducibility criterion Corollary 1.5(i) \Rightarrow (iii) was known for split groups over \mathbb{Q}_p due to the work of Frommer [Fro03], [Sch06, Proposition 2.6(ii)] and in general by Orlik–Strauch [OS10] (when $\dim_C \tau = 1$).

In a different direction, Ban–Hundley [BH16] argue on the dual side like [ST02] to prove the irreducibility of $(\mathrm{Ind}_B^G \chi)^{\mathrm{cts}}$ for split G and $|\chi|_C$ lying in a certain open cone, namely the cone where $e(\chi)$ (defined in §1.1) is *strictly* dominant. (This region excludes the unitary locus if G is not a torus.) Ban–Strauch [BS19] characterize the irreducibility of principal series of $\mathrm{SL}_n(F)$ in terms of the irreducibility of principal series of $\mathrm{GL}_n(F)$.

Finally, in the unitary case weak results can be obtained from smooth mod p representation theory, by using that an admissible unitary Banach space representation is irreducible provided its reduction modulo a G -stable open and bounded lattice is irreducible as smooth representation. The main results of our earlier work [AHHV17] then show that if σ is unitary and the reduction $\bar{\sigma}|_{Z \cap L'_\alpha}$ of $\sigma|_{Z \cap L'_\alpha}$ is non-trivial for all simple roots α , then $(\mathrm{Ind}_B^G \sigma)^{\mathrm{cts}}$ is irreducible.

1.9. Notation. Let C be a finite extension of \mathbb{Q}_p with uniformizer ϖ_C . In this paper, unless otherwise stated, the coefficient field of any representation is C .

Let F be a finite extension of \mathbb{Q}_p contained in C and \underline{G} a connected reductive group over F , $Z_{\underline{G}}$ the center of \underline{G} , $G = \underline{G}(F)$ is the group of rational points, \mathfrak{g} the Lie algebra of \underline{G} , $\mathfrak{g}_C := \mathfrak{g} \otimes_F C$ and $U(\mathfrak{g}_C)$ the enveloping algebra of \mathfrak{g}_C . We use the same notation for other groups. For simplicity we assume that \underline{G} splits over C . Moreover, starting in subsection 2.3 we will assume that C be sufficiently large, depending only on \underline{G} . Let $\underline{S} \subset \underline{G}$ be a maximal split torus and \underline{Z} the centralizer of \underline{S} in \underline{G} . This is a Levi subgroup of a minimal parabolic subgroup \underline{B} , which we fix from now on, and we let \underline{U} denote its unipotent radical.

We fix a special point in the apartment of \underline{S} and let K be the corresponding special parahoric subgroup.

Suppose $P = \underline{L}N$ is a parabolic subgroup of \underline{G} with Levi part \underline{L} and unipotent radical \underline{N} . For a smooth representation σ of $L = \underline{L}(F)$, let $(\mathrm{Ind}_P^G \sigma)^{\mathrm{sm}}$ be the smooth parabolically induced representation, namely the space of locally constant functions

$f: G \rightarrow \sigma$ such that $f(\ell ng) = \sigma(\ell)f(g)$ for $\ell \in L$, $n \in N$ and $g \in G$. We use this notation even when σ is defined over a field different from C (for example, over the residue field of C or the field of complex numbers). Let δ_P be the modulus function of P defined by $\int_P f(xg)dx = \delta_P(g) \int_P f(x)dx$ for any integrable function f on P , where we use a left-invariant Haar measure. Let $(\mathrm{n}\text{-}\mathrm{Ind}_P^G \sigma)^{\mathrm{sm}} := (\mathrm{Ind}_P^G \sigma \delta_P^{1/2})^{\mathrm{sm}}$ be the normalized induced representation.

We say that a continuous representation of a topological group is *irreducible* if it is topologically irreducible.

1.10. Acknowledgements. We are very grateful to Jean-Loup Waldspurger for sending us a proof of Proposition 3.7 and for allowing us to reproduce it here, and to Alberto Mínguez for his help with Proposition 3.22 as well as for useful discussions. We thank Tasho Kaletha for help with the proof of Proposition A.3. We are thankful to Sascha Orlik and Matthias Strauch for helpful conversations about their work and for allowing us to include our appendix. The debt this paper owes to the work of Orlik–Strauch should be evident to the reader. We thank Hiraku Atobe, Laurent Clozel, and Sug Woo Shin for helpful comments. Part of this work was done during a pleasant stay of the first-named author at University of Toronto.

2. IRREDUCIBILITY CRITERION

2.1. Banach representations. A representation π of G is called a *Banach representation* if π is a Banach space and the action map $G \times \pi \rightarrow \pi$ is continuous. The notion of *admissible* Banach representations was introduced by Schneider–Teitelbaum [ST02]. Any morphism between admissible Banach representations is strict and, in particular, has a closed image. By [ST02, Theorem 3.5] admissible Banach representations of G form an artinian abelian category, anti-equivalent (for any fixed compact open subgroup H of G) to the category of $\mathcal{O}_C[[H]][\frac{1}{p}]$ -modules that have a compatible action of G . Since it is an abelian category, we have notions such as finite length objects (traditionally called admissible Banach representations that are topologically of finite length). The subobjects (resp. quotient objects) are precisely the closed G -subrepresentations (resp. Hausdorff quotient representations), using for example [ST02, Proposition 1.3]. Admissible representations satisfy “Schur’s lemma”, namely if π is an admissible irreducible Banach representation, then the C -algebra of continuous G -endomorphisms $\mathrm{End}_G^{\mathrm{cts}}(\pi)$ is finite-dimensional [DS13, Theorem 1.1]. In particular, if π is absolutely irreducible (i.e. irreducible after any finite extension of scalars), then $\mathrm{End}_G^{\mathrm{cts}}(\pi) = C$ and π therefore has a central character.

Let \underline{P} be a semistandard parabolic subgroup, \underline{N} the unipotent radical of \underline{P} , \underline{L} a Levi part containing \underline{S} (hence $P = LN$ and $P \cap K = (L \cap K)(N \cap K)$) and σ a Banach representation of L . Then the representation $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is defined as the space of continuous functions $f: G \rightarrow \sigma$ such that $f(\ell ng) = \sigma(\ell)f(g)$ for any $g \in G$, $\ell \in L$ and $n \in N$. It is naturally a closed subspace of $\mathcal{C}^0(G, \sigma)$ equipped with the compact open topology, on which G acts continuously [Eme17, Proposition 3.1.5]. The isomorphism $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}} \cong (\mathrm{Ind}_{P \cap K}^K \sigma)^{\mathrm{cts}}$ shows that it is a Banach space. It is admissible if σ is admissible [Fu21, Lemma 3.3]. By [Eme17, Lemma 6.5.5] the topology of σ can be defined by an $L \cap K$ -invariant norm $|\cdot|$. On the right-hand side of the isomorphism $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}} \cong (\mathrm{Ind}_{P \cap K}^K \sigma)^{\mathrm{cts}}$ we have a K -invariant norm $\|\cdot\|$ defined by $\|f\| = \sup_{x \in K/(P \cap K)} |f(x)|$, which defines the topology. Its unit ball is $((\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}})^0 = \{f \in (\mathrm{Ind}_{P \cap K}^K \sigma)^{\mathrm{cts}} \mid f(K) \subset \sigma^0\} = (\mathrm{Ind}_{P \cap K}^K \sigma^0)^{\mathrm{cts}}$ and $((\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}})^0 / \varpi_C((\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}})^0 \cong (\mathrm{Ind}_{P \cap K}^K (\sigma^0 / \varpi_C \sigma^0))^{\mathrm{sm}}$ and this is a smooth representation of K . If $\pi \subset (\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is a closed subspace, then

it is a Banach space with the induced norm, and the natural map $\pi^0/\varpi_C\pi^0 \rightarrow ((\text{Ind}_P^G \sigma)^{\text{cts}})^0/\varpi_C((\text{Ind}_P^G \sigma)^{\text{cts}})^0$ is injective.

2.2. Locally analytic representations. Let π be a Banach representation of G . We say that $v \in \pi$ is an $(F\text{-})$ locally analytic vector if the map $G \ni g \mapsto gv \in \pi$ is $(F\text{-})$ locally analytic. The space of locally analytic vectors in π is denoted by π^{an} . We impose a topology on π^{an} as in [Eme17, Definition 3.5.3]. Then π^{an} is a locally analytic representation in the sense that π^{an} is a barrelled locally convex Hausdorff vector space and the map $G \ni g \mapsto gv \in \pi^{\text{an}}$ is locally analytic for any $v \in \pi^{\text{an}}$. We have a notion of *admissible* locally analytic representations introduced by Schneider–Teitelbaum [ST03, page 176], and the category of such representations is abelian [ST03, Proposition 6.4].

Theorem 2.1 ([ST03, Theorem 7.1], [Eme17, Proposition 6.2.4]). *Suppose that $F = \mathbb{Q}_p$. Then the functor $(\cdot)^{\text{an}}$ sends admissible Banach representations of G to admissible locally analytic representations of G on compact type spaces. It is moreover exact. If π is an admissible Banach representation, then π^{an} is dense in π .*

For proving density, the following criterion is sometimes useful. A more general version can be found in [AH, §2.2]. Let \overline{N} denote the unipotent radical of the opposite parabolic subgroup $\overline{P} = \underline{L}\overline{N}$.

Lemma 2.2. *Let σ be an irreducible Banach representation of L with a central character and $0 \neq f \in (\text{Ind}_P^G \sigma)^{\text{cts}}$. Assume that $\text{supp}(f) \subset P \setminus P\overline{N}$ and there exists $v \in \sigma$ such that $f(\overline{n}) \in Cv$ for all $\overline{n} \in \overline{N}$. Then f is a topological generator of $(\text{Ind}_P^G \sigma)^{\text{cts}}$, i.e. the subspace spanned by $\{gf \mid g \in G\}$ is dense in $(\text{Ind}_P^G \sigma)^{\text{cts}}$.*

Here, $\text{supp}(f)$ denotes the *closure* of the set $P \setminus \{x \in G : f(x) \neq 0\}$ in $P \setminus G$. In particular, $\text{supp}(f)$ is compact.

Proof. We choose an $L \cap K$ -invariant norm $|\cdot|$ on σ that defines its topology. We may assume $\{|v'| \mid v' \in \sigma\} = |C^\times| \cup \{0\}$ and $|v| = 1$.

Let \overline{N}_0 be a compact open subgroup of $\overline{N} \cap K$. Let π be the closure of the subspace spanned by $\{gf \mid g \in G\}$ in $(\text{Ind}_P^G \sigma)^{\text{cts}}$. By the action of the center of L there exists $f' \in \pi$ such that $\text{supp}(f') \subset P\overline{N}_0$ and $f'(\overline{n}) \in Cv$ for any $\overline{n} \in \overline{N}$. We may scale f' such that $\|f'\| = 1$. Hence there exists $f' \in \pi$ such that $\|f'\| = 1$, $\text{supp}(f') \subset P\overline{N}_0$ and $f'(\overline{n}) \in Cv$ for any $\overline{n} \in \overline{N}$.

We will now prove that $\pi = (\text{Ind}_P^G \sigma)^{\text{cts}}$. Let X_0 be the closed subspace of $h \in (\text{Ind}_P^G \sigma)^{\text{cts}}$ such that $\text{supp}(h) \subset P\overline{N}_0$ and $h(\overline{n}) \in Cv$ for any $\overline{n} \in \overline{N}$. Let $C^0(\overline{N}_0, Cv)$ be the Banach space of continuous functions $\overline{N}_0 \rightarrow Cv$ equipped with the supremum norm. The restriction to \overline{N}_0 gives an \overline{N}_0 -equivariant norm-preserving isomorphism $X_0 \cong C^0(\overline{N}_0, Cv)$ of Banach spaces.

We have an embedding $(X_0 \cap \pi)^0/\varpi_C(X_0 \cap \pi)^0 \hookrightarrow X_0^0/\varpi_C X_0^0$ and from the second paragraph the image is nonzero. Since the image is a smooth \overline{N}_0 -representation, it contains a nonzero \overline{N}_0 -fixed vector. Let $\overline{v} \in \sigma^0/\varpi_C \sigma^0$ be the image of v . The space of \overline{N}_0 -fixed vectors in $X_0^0/\varpi_C X_0^0 \cong C^\infty(\overline{N}_0, (\mathcal{O}_C/\varpi_C \mathcal{O}_C)\overline{v})$ is one-dimensional and spanned by the constant function $1_{\overline{N}_0}$.

Let X be the space of $h \in (\text{Ind}_P^G \sigma)^{\text{cts}}$ such that $\text{supp}(h) \subset P(\overline{N} \cap K)$ and $h(\overline{n}) \in Cv$ for any $\overline{n} \in \overline{N} \cap K$. Then $X^0/\varpi_C X^0 \cong C^\infty(\overline{N} \cap K, (\mathcal{O}_C/\varpi_C \mathcal{O}_C)\overline{v})$ is spanned by $\{1_{\overline{N}_0} \mid \overline{N}_0 \subset \overline{N} \cap K \text{ is an open subgroup}\}$ as $\overline{N} \cap K$ -representation. Since each $1_{\overline{N}_0}$ is in $(\pi \cap X)^0/\varpi_C(\pi \cap X)^0$ as we have proved, we have $(\pi \cap X)^0/\varpi_C(\pi \cap X)^0 = X^0/\varpi_C X^0$. In other words, for any $h \in X^0$, there exists $f_0 \in (\pi \cap X)^0$ such that $h - f_0 \in \varpi_C X^0$. By iterating this argument, for each $k = 0, 1, \dots$ there exists $f_k \in \pi$ such that $h - f_k \in \varpi_C^k X^0$. Then $h = \lim_{k \rightarrow \infty} f_k \in \pi$. This shows that

$X^0 \subset \pi$, hence $X \subset \pi$. By the action of the center of L , for any continuous function $\varphi: \overline{N} \rightarrow C$ with compact support, the function $\varphi \otimes v: \overline{N} \rightarrow C^0 v$ defined by $(\varphi \otimes v)(\overline{n}) = \varphi(\overline{n})v$ is in π . Here we regard $\varphi \otimes v$ as element of $(\text{Ind}_P^G \sigma)^{\text{cts}}$ (supported on $P\overline{N}$).

Let $\ell \in L$ and define $\ell\varphi: \overline{N} \rightarrow C$ by $(\ell\varphi)(\overline{n}) = \varphi(\ell^{-1}\overline{n}\ell)$. For each $\varphi: \overline{N} \cap K \rightarrow C$ and $\ell \in L$ there exists z in the center of L such that $z\ell^{-1}\varphi$ is supported on $\overline{N} \cap K$. We have $\varphi \otimes \ell v = \omega_\sigma(z)\ell z^{-1}(z\ell^{-1}\varphi \otimes v)$, where ω_σ is the central character of σ . Since $z\ell^{-1}\varphi \otimes v \in \pi$ by the previous paragraph, we have $\varphi \otimes \ell v \in \pi$. Since σ is irreducible, we have $\varphi \otimes v' \in \pi$ for any $\varphi \in C^0(\overline{N} \cap K, C)$ and $v' \in \sigma$. In other words, $C^0(\overline{N} \cap K, C) \otimes \sigma \subset \pi$.

Let $h: \overline{N} \cap K \rightarrow \sigma^0$ be any continuous function and for each $k = 0, 1, \dots$ define h_k by $\overline{N} \cap K \xrightarrow{h} \sigma^0 \rightarrow \sigma^0 / \varpi_C^k \sigma^0$. Then h_k is a locally constant function and therefore by compactness has finite image in $\sigma^0 / \varpi_C^k \sigma^0$. Hence there exists $h'_k \in C^0(\overline{N} \cap K, C) \otimes \sigma$ (which we can even take to be locally constant) such that $h - h'_k \in \varpi_C^k C^0(\overline{N} \cap K, \sigma^0)$. Therefore $h = \lim_{k \rightarrow \infty} h'_k$. Regarding h as element of $(\text{Ind}_P^G \sigma)^{\text{cts}}$, the previous paragraph shows that $h \in \pi$. Since such h generate $(\text{Ind}_P^G \sigma)^{\text{cts}}$ as a G -representation, we get $\pi = (\text{Ind}_P^G \sigma)^{\text{cts}}$. \square

If σ is a locally analytic representation of P on a locally convex topological vector space of compact type, we denote by $(\text{Ind}_P^G \sigma)^{\text{an}}$ the space of locally analytic functions $G \rightarrow \sigma$ such that $f(p'g) = \sigma(p')f(g)$ for all $g \in G$, $p' \in P$. This is a locally analytic representation of G of compact type, see e.g. [BH20, §2]. (Usually, σ will arise by inflation from a locally analytic representation of L .)

Lemma 2.3.

- (i) *If τ is a locally analytic representation of P on a compact type space and V a finite-dimensional locally analytic representation of G , then $V \otimes (\text{Ind}_P^G \tau)^{\text{an}} \cong (\text{Ind}_P^G V \otimes \tau)^{\text{an}}$ as locally analytic representations of G .*
- (ii) *If τ is a Banach representation of P and V a finite-dimensional continuous representation of G , then $V \otimes (\text{Ind}_P^G \tau)^{\text{cts}} \cong (\text{Ind}_P^G V \otimes \tau)^{\text{cts}}$ as Banach representations of G .*
- (iii) *If τ is a Banach representation of P , then $((\text{Ind}_P^G \tau)^{\text{cts}})^{\text{an}} \cong (\text{Ind}_P^G \tau^{\text{an}})^{\text{an}}$ as locally analytic representations of G .*
- (iv) *If π is a Banach representation of G and V a finite-dimensional locally analytic representation of G , then $(V \otimes \pi)^{\text{an}} \cong V \otimes \pi^{\text{an}}$ as locally analytic representations of G .*
- (v) *If π is an admissible Banach representation of G and V a finite-dimensional continuous representation of G , then $V \otimes \pi$ is an admissible Banach representation of G .*

Note that the tensor products carry the projective, or equivalently inductive, topology.

Proof. (i) We have a natural map of locally analytic representations

$$\begin{aligned} \Theta: V \otimes (\text{Ind}_P^G \tau)^{\text{an}} &\rightarrow (\text{Ind}_P^G V \otimes \tau)^{\text{an}} \\ v \otimes f &\mapsto (g \mapsto gv \otimes f(g)). \end{aligned}$$

Let $(v_i)_{i=1}^n$ be a basis of V . Suppose that $h \in (\text{Ind}_P^G V \otimes \tau)^{\text{an}}$ and write $h(g) = \sum_i gv_i \otimes f_i(g)$ for unique P -equivariant functions $f_i: G \rightarrow \tau$. It suffices to show that the functions f_i are locally analytic. Let $(v_i^*)_{i=1}^n$ denote the dual basis of V and let $\lambda_j: G \rightarrow V^\vee$ denote the locally analytic function $\lambda_j(g) := gv_j^*$. Then f_j is obtained from h and λ_j via the natural bilinear pairing $(V \otimes \tau) \times V^\vee \rightarrow \tau$, so it is

locally analytic by [FdL99, Satz 2.4.3]. (The assumption BIL there is satisfied, as V is finite-dimensional.)

(ii) The proof is analogous but easier, using that the bilinear pairing above is continuous.

(iii) is proved in [Fu21, Theorem 3.6], (iv) is proved in [Eme17, Proposition 3.6.15] (more generally), and (v) is proved in [Eme17, Proposition 6.2.6]. \square

Let G' denote the smallest normal subgroup of G generated by U . It is a closed locally analytic subgroup of G [BT73, §6].

Lemma 2.4.

- (i) *Any locally analytic character $\psi: G \rightarrow C^\times$ and any finite-dimensional smooth representation τ of G is trivial on G' .*
- (ii) *Suppose that $\underline{P} = \underline{LN}$ is a parabolic subgroup and that τ is a finite-dimensional continuous representation of P . Then $\tau^N \neq 0$. In particular, if τ is irreducible, then τ is trivial on N .*

Proof. (ii) We may assume that $F = \mathbb{Q}_p$, so τ is a locally analytic representation of P . By finite-dimensional representation theory of (a Borel subalgebra of) \mathfrak{p}_C , we have $\tau^{\mathfrak{n}_C} \neq 0$. This is a P -subrepresentation of τ on which N acts smoothly. By using the S -action, we see that $\tau^{\mathfrak{n}_C} = \tau^N$.

(i) The claim for ψ follows from (ii), as G' is the normal subgroup generated by U , and the argument for smooth τ is similar to (ii) (but easier and well-known). \square

2.3. The functor of Orlik–Strauch. In this section we discuss the categories $\mathcal{O}^{\mathfrak{p}}$, \mathcal{O}^P and functors \mathcal{F}_P^G of Orlik–Strauch for general \underline{G} . We also take up and extend some general notions from [Or18].

Let $\underline{P} = \underline{LN}$ be a standard parabolic subgroup. Recall the abelian categories $\mathcal{O}^{\mathfrak{p}}$ and \mathcal{O}^P . First, $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $U(\mathfrak{g}_C)$ -modules M such that

- (i) M is a finitely generated $U(\mathfrak{g}_C)$ -module,
- (ii) M is a direct sum of absolutely simple finite-dimensional \mathfrak{l}_C -modules,
- (iii) the action of \mathfrak{p}_C on M is locally finite.

The objects of \mathcal{O}^P consist of pairs (M, τ) , where $M \in \mathcal{O}^{\mathfrak{p}}$ and τ is a locally finite-dimensional locally analytic action of P on M whose derivative equals the given action of $\mathfrak{p}_C \subset \mathfrak{g}_C$ and such that $g \circ X \circ g^{-1} = \text{Ad}(g)(X)$ on M for all $g \in P$, $X \in U(\mathfrak{g}_C)$. The morphisms consist of maps in $\mathcal{O}^{\mathfrak{p}}$ that are moreover P -linear.

In particular, \mathcal{O}^G consists of all finite-dimensional locally analytic representations of G on which the derived action of \mathfrak{g}_C is a direct sum of absolutely simple \mathfrak{g}_C -modules. This category is stable under duality. Note that if τ is a finite-dimensional smooth representation of P , then $\tau \in \mathcal{O}^P$ (killed by \mathfrak{g}_C). We also remark that any object in $\mathcal{O}^{\mathfrak{p}}$, \mathcal{O}^P is of finite length [Hum08, 1.11 Theorem].

Let \underline{T}' be a maximal split torus containing \underline{S}_C in the split group \underline{Z}_C and let \mathfrak{t}' be its Lie algebra. Let \mathfrak{z}_L denote the Lie algebra of the center \underline{Z}_L of \underline{L} .

Lemma 2.5. *Condition (ii) in the definition of category $\mathcal{O}^{\mathfrak{p}}$ may be replaced by either*

- (ii') *M is a direct sum of 1-dimensional \mathfrak{t}' -modules; or*
- (ii'') *M is a direct sum of 1-dimensional $\mathfrak{z}_{L,C}$ -modules.*

Proof. This is a question about \mathfrak{g}_C -modules, so we may work over $F = C$. In particular, \underline{G} is split and $\underline{T}' = \underline{S}$ (and we can drop all extensions of scalars to C). Note that (ii') implies (ii''), as $\mathfrak{z}_L \subset \mathfrak{t}'$. To see that (ii) implies (ii'), let W be any absolutely simple finite-dimensional \mathfrak{l} -module. Then $W^{\mathfrak{u}}$ is 1-dimensional, so the surjection $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} W^{\mathfrak{u}} \twoheadrightarrow W$ shows that W is a direct sum of 1-dimensional \mathfrak{t}' -modules.

Finally we show that (ii'') implies (ii). We know that M is a sum of finite-dimensional \mathfrak{l} -submodules by (iii). It suffices to show that any finite-dimensional \mathfrak{l} -module W on which \mathfrak{z}_L acts diagonalizably is a sum of absolutely simple \mathfrak{l} -modules. Write $\mathfrak{l} = \mathfrak{l}^{\text{der}} \oplus \mathfrak{z}_L$. By the diagonalizability, W is a direct sum of \mathfrak{z}_L -isotypic components, and this decomposition is preserved by $\mathfrak{l}^{\text{der}}$. As $\mathfrak{l}^{\text{der}}$ is a semisimple Lie algebra we may assume that W is simple. By highest weight theory, any simple module of $\mathfrak{l}^{\text{der}} \otimes \overline{C}$ may be defined over C , hence W is absolutely simple. \square

Remark 2.6. Note that for any $M, M' \in \mathcal{O}^P$ with M' finite-dimensional the tensor product $M \otimes M'$ lies in \mathcal{O}^P as well. (Use condition (ii').)

Remark 2.7. Suppose that V is a finite-dimensional locally analytic representation of a parabolic subgroup $P = LN$. From the locally analytic homomorphism $P \rightarrow \text{GL}(V)$ and the functoriality of the logarithm map [Bou72, III.7.6] we see that

$$(2.1) \quad nv = \sum_{i=0}^{\infty} \frac{(\log n)^i}{i!} v$$

for all $n \in N$ near 1 and all $v \in V$. By using the action of L we see that in fact formula (2.1) is valid for all $n \in N$. We will just say that we “integrate” over N .

Lemma 2.8. *Suppose $P \subset Q$.*

- (i) *The forgetful functor $\mathcal{O}^Q \rightarrow \mathcal{O}^P$ is fully faithful.*
- (ii) *If $M \in \mathcal{O}^Q$ and $M' \in \mathcal{O}^P$ is a subquotient, then $M' \in \mathcal{O}^Q$.*
- (iii) *If $M \in \mathcal{O}^Q$ is simple, it is simple in \mathcal{O}^P as well.*

Proof. (i) Let $M_1, M_2 \in \mathcal{O}^Q$ and we prove that any morphism $\varphi: M_1 \rightarrow M_2$ in \mathcal{O}^P is Q -equivariant. Since φ is $\overline{u}_C \cap \mathfrak{q}_C$ -equivariant and this subalgebra acts locally nilpotently on M_1, M_2 , by integration, φ is $\overline{U} \cap Q$ -equivariant. Since Q is generated by P and $\overline{U} \cap Q$, φ is Q -equivariant.

(ii) We may assume M' is a subobject of M in \mathcal{O}^P . As in (i), by integration, M' is $\overline{U} \cap Q$ -stable and hence Q -stable. This implies (iii). \square

Remark 2.9. If $\varphi: \underline{G}_1 \rightarrow \underline{G}$ is a morphism such that $\varphi(\underline{G}_1^{\text{der}}) = \underline{G}^{\text{der}}$, $\ker \varphi \subset \underline{Z}_{\underline{G}_1}$, and the parabolic $\underline{P}_1 = \varphi^{-1}(\underline{P})$ is obtained as pre-image of \underline{P} , then we obtain functors $\mathcal{O}^{\mathfrak{p}} \rightarrow \mathcal{O}^{\mathfrak{p}_1}$ and $\mathcal{O}^P \rightarrow \mathcal{O}^{P_1}$ by inflation, which will occasionally be useful.

Lemma 2.10. *Any simple object of $\mathcal{O}^{\mathfrak{p}}$ is absolutely simple. In particular, any \mathfrak{g}_C -simple object of \mathcal{O}^P is absolutely simple.*

Proof. This is a question about \mathfrak{g}_C -modules, so we may work over $F = C$. In particular, \underline{G} is split (and we can drop all extensions of scalars to C). Let $W \in \mathcal{O}^{\mathfrak{p}}$ be simple. As in the proof of Lemma 2.5 we can take $\lambda \hookrightarrow W^{\mathfrak{u}}$ a 1-dimensional \mathfrak{t}' -submodule, so $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \lambda \twoheadrightarrow W$. Therefore the weight space W_λ is 1-dimensional and generates W , hence $\text{End}_{\mathfrak{g}}(W) = C$, i.e. W is absolutely simple. \square

Let $W \in \mathcal{O}^L$. Then on the generalized Verma module $\underline{M}(W) = \underline{M}_G(W) := U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W$, P acts by $p'(X \otimes w) = \text{Ad}(p')X \otimes p'w$ for $p' \in P$, $w \in W$ and $X \in \mathfrak{g}_C$. We have $\underline{M}(W) \in \mathcal{O}^P$. (More generally, this construction works if W is a finite-dimensional locally analytic representation of P on which \mathfrak{l}_C is a direct sum of absolutely simple \mathfrak{l}_C -modules, but this is not traditionally called a generalized Verma module.) If $W \in \mathcal{O}^L$ is simple, then $\underline{M}(W)$ has a unique simple quotient in \mathcal{O}^P (by the lemma that follows), which we denote by $\underline{L}(W) = \underline{L}_G(W)$.

Occasionally we will also use the generalized Verma module $M(W) := U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W$ for $W \in \mathcal{O}^{\mathfrak{l}}$ and its simple quotient $L(W)$ (if W is simple).

Lemma 2.11. *If $M \in \mathcal{O}^P$ is simple (resp. absolutely simple), then $M^N \in \mathcal{O}^L$ is simple (resp. absolutely simple). Moreover, $\underline{M}(M^N)$ has M as unique simple quotient in \mathcal{O}^P .*

Conversely, if $M' \in \mathcal{O}^L$ is simple (resp. absolutely simple), then $\underline{M}(M')$ has a unique simple (resp. absolutely simple) quotient Q in \mathcal{O}^P , and $Q^N \cong M'$. Moreover, Q is the largest semisimple quotient of $\underline{M}(M')$ in \mathcal{O}^P .

Proof. Pick any nonzero submodule $M' \subset M^N = M^{\mathfrak{n}_C}$ in \mathcal{O}^L , so $\underline{M}(M') = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} M' \twoheadrightarrow M$. As M is simple, $M|_{\mathfrak{g}_C}$ is semisimple, so $M|_{\mathfrak{g}_C} \cong \bigoplus_V V \otimes M_V$, where V runs through isomorphism classes of simple modules in \mathcal{O}^P and M_V is the multiplicity space. Note that $M^{\mathfrak{n}_C} \cong \bigoplus_V V^{\mathfrak{n}_C} \otimes M_V$ with $V^{\mathfrak{n}_C} \in \mathcal{O}^L$ (absolutely) simple \mathfrak{l}_C -modules. Moreover by highest weight theory we see that $V_1 \cong V_2$ if and only if $V_1^{\mathfrak{n}_C} \cong V_2^{\mathfrak{n}_C}$ for simple modules $V_1, V_2 \in \mathcal{O}^P$, i.e. $M^{\mathfrak{n}_C} \cong \bigoplus_V V^{\mathfrak{n}_C} \otimes M_V$ is the decomposition into isotypic components. Since M' is an \mathfrak{l}_C -submodule of $M^{\mathfrak{n}_C}$, $M' \cong \bigoplus_V V^{\mathfrak{n}_C} \otimes M'_V$ for some subspaces $M'_V \subset M_V$. Therefore $M' = W^{\mathfrak{n}_C}$, where $W := \bigoplus_V V \otimes M'_V$ (considered as \mathfrak{g}_C -submodule of M). Now the image of $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} M' \twoheadrightarrow M$ is contained in W , so $W = M$. Hence $M' = M^N$, i.e. M^N is simple. The absolutely simple case follows.

If $M' \in \mathcal{O}^L$ is simple, then $M'|_{\mathfrak{l}_C} \cong \bigoplus_{i=1}^r W'_i$ with W'_i absolutely simple \mathfrak{l}_C -modules. Then $\underline{M}(M') = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} M'$ has a largest semisimple quotient Q as \mathfrak{g}_C -module, and this quotient is in \mathcal{O}^P , as the \mathfrak{g}_C -radical is P -stable. Moreover the composition $M' \hookrightarrow \underline{M}(M') \twoheadrightarrow Q$ has image $Q^{\mathfrak{n}_C}$, so $M' \cong Q^N$ as L -representations. Let $Q' \subset Q$ denote a simple subobject. Then $0 \neq Q'^N \subset Q^N$, so $Q'^N = Q^N$. By construction, Q is a direct sum of absolutely simple \mathfrak{g}_C -modules, hence no proper \mathfrak{g}_C -submodule of Q can have the same \mathfrak{n}_C -invariants as Q , so $Q' = Q$, i.e. Q is simple in \mathcal{O}^P . The absolutely simple case follows, as our construction of Q commutes with scalar extensions.

Going back to the first part, $\underline{M}(M')$ surjects onto M , hence by the previous paragraph M is its unique simple quotient. \square

Corollary 2.12. *Suppose that $W \in \mathcal{O}^L$ is simple and that $Q = L_Q N_Q$ is a parabolic subgroup containing P . Then $\underline{L}(W)^{N_Q} \cong \underline{L}_{L_Q}(W)$ in $\mathcal{O}^{P \cap L_Q}$. Moreover, we have $\underline{L}(W) \in \mathcal{O}^Q$ if and only if $\underline{L}_{L_Q}(W) \in \mathcal{O}^{L_Q}$, and in this case $\underline{L}(W) \cong \underline{L}(\underline{L}_{L_Q}(W))$ in \mathcal{O}^Q .*

Proof. We first note that if $M \in \mathcal{O}^P$, then $M^{\mathfrak{n}_C} \in \mathcal{O}^{P \cap L_Q}$. Hence if $M \in \mathcal{O}^P$, then $M^{N_Q} = M^{\mathfrak{n}_C} \in \mathcal{O}^{P \cap L_Q}$. We claim that if $M \in \mathcal{O}^P$ is simple, then M^{N_Q} is simple in $\mathcal{O}^{P \cap L_Q}$. Suppose by contradiction that $0 \rightarrow M' \rightarrow M^{N_Q} \rightarrow M'' \rightarrow 0$ in $\mathcal{O}^{P \cap L_Q}$ with M', M'' nonzero. Then the sequence splits as $\mathfrak{l}_{Q,C}$ -modules, so it remains exact on $\mathfrak{n}_C \cap \mathfrak{l}_{Q,C}$ -invariants. In other words, $0 \rightarrow (M')^{N \cap L_Q} \rightarrow M^N \rightarrow (M'')^{N \cap L_Q} \rightarrow 0$ is exact in \mathcal{O}^L . On the other hand, $(M')^{N \cap L_Q} = (M')^{\mathfrak{n}_C \cap \mathfrak{l}_{Q,C}} \neq 0$ and likewise for $(M'')^{N \cap L_Q}$, contradicting the simplicity of M^N in \mathcal{O}^L (Lemma 2.11). This proves the claim. The isomorphism $\underline{L}(W)^{N_Q} \cong \underline{L}_{L_Q}(W)$ in $\mathcal{O}^{P \cap L_Q}$ follows by taking $M := \underline{L}(W)$ and applying Lemma 2.11.

In particular, if $\underline{L}(W) \in \mathcal{O}^Q$, then $\underline{L}_{L_Q}(W) \in \mathcal{O}^{L_Q}$. Conversely, if $\underline{L}_{L_Q}(W) \in \mathcal{O}^{L_Q}$ then $\underline{L}(\underline{L}_{L_Q}(W)) \in \mathcal{O}^Q$ and by taking N -invariants in stages we can identify $\underline{L}(\underline{L}_{L_Q}(W))$ with $\underline{L}(W)$ in \mathcal{O}^P , and hence in \mathcal{O}^Q , by Lemma 2.11. \square

Lemma 2.13. *Suppose that $W \in \mathcal{O}^L$ and $X \in \mathcal{O}^P$ with $X = X^N$. Then $W \otimes X \in \mathcal{O}^L$ and $\underline{M}(W \otimes X) \cong \underline{M}(W) \otimes X$ in \mathcal{O}^P . If moreover $W \otimes X$ is simple in \mathcal{O}^L and $\underline{L}(W) \otimes X$ is simple in \mathcal{O}^P , then $\underline{L}(W \otimes X) \cong \underline{L}(W) \otimes X$ in \mathcal{O}^P .*

In particular, X could be a locally analytic character of G (cf. Lemma 2.4).

Proof. We have $X = X^N \in \mathcal{O}^L$, so $W \otimes X \in \mathcal{O}^L$ by Remark 2.6. The natural isomorphism $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} (W \otimes X) \rightarrow (U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W) \otimes X$ of \mathfrak{g}_C -modules sending $1 \otimes (w \otimes x)$ to $(1 \otimes w) \otimes x$ (cf. [Hum08, §3.6]) is also P -equivariant, which completes the proof of the first claim. The second claim follows by Lemma 2.11 (noting that W is simple, as $W \otimes X$ is simple). \square

Let $\text{Rep}^{\text{adm}}(L)$ be the category of admissible smooth L -representations. When \underline{G} is split, Orlik–Strauch [OS14b] define a functor \mathcal{F}_P^G from $(\mathcal{O}^P)^{\text{op}} \times \text{Rep}^{\text{adm}}(L)$ to the category of admissible locally analytic representations of G . Their functor can be generalized to any \underline{G} , see Appendix A.

For the theory of Orlik–Strauch, we need a small assumption on p as follows [OS14b, Assumption 4.1].

Assumption 2.14. *If the absolute root system of \underline{G} has irreducible components of type B, C or F_4 , we assume $p > 2$. If the absolute root system of \underline{G} has irreducible components of type G_2 , we assume $p > 3$.*

Definition 2.15. We say that $M \in \mathcal{O}^P$ is *equimaximal* if for any parabolic subgroup Q containing P we have $M \in \mathcal{O}^Q$ if and only if $M|_{\mathfrak{g}_C} \in \mathcal{O}^{\mathfrak{q}}$. In this case we say that M has *maximal parabolic* Q if Q is largest among all parabolic subgroups containing P such that $M \in \mathcal{O}^Q$.

Recall that we assumed in §1.9 that C be sufficiently large. The reason is part (iii) of the following theorem (see Appendix A).

Theorem 2.16 (Theorem A.1). *Assume Assumption 2.14.*

- (i) *The functor \mathcal{F}_P^G is exact in both arguments.*
- (ii) *Let $Q = L_Q N_Q \supset P$ be another parabolic subgroup. If $M \in \mathcal{O}^Q$ and $\pi \in \text{Rep}^{\text{adm}}(L)$, then $\mathcal{F}_P^G(M, \pi) \cong \mathcal{F}_Q^G(M, (\text{Ind}_{P \cap L_Q}^{L_Q} \pi)^{\text{sm}})$.*
- (iii) *Assume that $M \in \mathcal{O}^P$ is equimaximal with maximal parabolic P and $\pi \in \text{Rep}^{\text{adm}}(L)$. Assume that $M|_{\mathfrak{g}_C} \in \mathcal{O}^{\mathfrak{p}}$ is simple and π is irreducible. Then $\mathcal{F}_P^G(M, \pi)$ is irreducible.*

We also note two more basic properties that follow by construction. For a locally convex space V , let V' be the strong dual space of V , namely V' is the space of continuous linear maps $V \rightarrow C$ with the topology of uniform convergence on bounded subsets of V .

Proposition 2.17.

- (i) *Suppose that W is a finite-dimensional locally analytic representation of P on which \mathfrak{l}_C acts as a direct sum of absolutely simple \mathfrak{l}_C -modules and $\pi \in \text{Rep}^{\text{adm}}(L)$. Then $\mathcal{F}_P^G(\underline{M}(W), \pi) \cong (\text{Ind}_P^G W' \otimes \pi)^{\text{an}}$.*
- (ii) *If τ is any finite-dimensional smooth representation of L , $M \in \mathcal{O}^P$, $\pi \in \text{Rep}^{\text{adm}}(L)$, then $\mathcal{F}_P^G(M \otimes \tau, \pi) \cong \mathcal{F}_P^G(M, \pi \otimes \tau')$.*

The following corollary will be proved following Proposition 2.26 (which is needed as input).

Corollary 2.18. *Assume Assumption 2.14. Let $M \in \mathcal{O}^P$ and $\pi \in \text{Rep}^{\text{adm}}(L)$ such that π is of finite length. Then $\mathcal{F}_P^G(M, \pi)$ is strongly admissible and topologically of finite length.*

We say that a finite-dimensional representation of G (or more generally P) is *algebraic* if it is obtained by restriction from an algebraic representation of the split group $\underline{G}_C := G \times_F C$ (or \underline{P}_C).

We let $\mathcal{O}_{\text{alg}}^P \subset \mathcal{O}^P$ be the full subcategory consisting of those objects where the action of P is locally finite-dimensional algebraic. It is closed under subquotients.

The composition $\mathcal{O}_{\text{alg}}^P \rightarrow \mathcal{O}^P \rightarrow \mathcal{O}^{\mathfrak{p}}$ is fully faithful with essential image $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$, consisting of those objects where \mathfrak{t}' acts by elements of $X^*(\underline{T}') \subset \mathfrak{t}'^*$ [OS15, §2, Lemma 3.2]. If $W \in \mathcal{O}_{\text{alg}}^L$, then $\underline{M}(W) \in \mathcal{O}_{\text{alg}}^P$ and hence $\underline{L}(W) \in \mathcal{O}_{\text{alg}}^P$ (if W is simple). We state a useful consequence.

Lemma 2.19. *Any $M \in \mathcal{O}_{\text{alg}}^P$ is equimaximal.*

2.4. Some decompositions in \mathcal{O} . Recall that \underline{T}' is a maximal split torus containing \underline{S}_C in the split group \underline{Z}_C with Lie algebra \mathfrak{t}' . Let $\mathfrak{b}' = \mathfrak{t}' \oplus \mathfrak{u}'$ be a Borel subalgebra of \mathfrak{g}_C . We write $\tilde{\Phi}$ for the set of \mathfrak{t}' -roots in \mathfrak{g}_C and $\tilde{\Phi}^+$ for the set of \mathfrak{t}' -roots in \mathfrak{u}' . For $\lambda \in (\mathfrak{t}')^*$, the Verma module $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{b}')} \lambda$ has a unique irreducible quotient $L(\lambda)$ in $\mathcal{O}^{\mathfrak{b}'}$.

Lemma 2.20. *If H is a compact abelian locally F -analytic group, then any F -linear map $\lambda: \text{Lie } H \rightarrow C$ lifts to a locally analytic homomorphism $H \rightarrow C^\times$ (after perhaps replacing C by a finite extension). If \underline{H} is a torus over F , then any F -linear map $\lambda: \text{Lie } \underline{H} \rightarrow C$ lifts to a locally analytic homomorphism $\underline{H}(F) \rightarrow C^\times$ (after perhaps replacing C by a finite extension).*

Remark 2.21. The finite extension of C may depend on λ , as one can already see in case $H = \mathbb{Z}_p$ over $F = \mathbb{Q}_p$. (Note that if $\chi: \mathbb{Z}_p \rightarrow C^\times$, then $d\chi(\mathbb{Z}_p) \subset \log(\mathcal{O}_C^\times)$ is bounded.)

Proof. The given λ lifts to a locally analytic homomorphism $f: H_0 \rightarrow C^\times$ for some open subgroup H_0 of H by [Sch11, Proposition 18.17], so H_0 is of finite index. Using for example that $\mathcal{O}_{\overline{C}}^\times$ (where \overline{C} is an algebraic closure of C) is an injective abelian group we deduce that we can extend f to a homomorphism $H \rightarrow C'^\times$ for some finite extension C' of C .

For the second part, note that $\underline{H}(F) \cong \underline{H}(F)^1 \times \mathbb{Z}^d$ for some $d \geq 0$, where $\underline{H}(F)^1$ is the maximal compact subgroup [KP22, §2.5(b), (c)]. In particular, $\underline{H}(F)^1$ is a compact abelian locally F -analytic group and we can apply the first part. \square

Lemma 2.22. *Suppose that $\underline{G}^{\text{der}}$ is simply connected.*

- (i) *If \underline{G} is semisimple (simply-connected), then any simple object of $\mathcal{O}^{\mathfrak{g}}$ lifts to \mathcal{O}^G (even $\mathcal{O}_{\text{alg}}^G$). If moreover all simple factors of \underline{G} are isotropic, then $\mathcal{O}^{\mathfrak{g}} = \mathcal{O}_{\text{alg}}^G = \mathcal{O}^G$.*
- (ii) *In general, any simple object of $\mathcal{O}^{\mathfrak{p}}$ lifts to \mathcal{O}^P , after perhaps replacing C by a finite extension. Any two lifts differ by a smooth character of L .*
- (iii) *If M is a \mathfrak{g}_C -simple object in \mathcal{O}^G , then $M \cong M_{\text{alg}} \otimes \psi$, where M_{alg} is an algebraic representation of G and ψ a locally analytic character of G (after perhaps replacing C by a finite extension).*

Remark 2.23. The second claim in (i) fails without the condition on the simple factors. For example, let $G = \text{SL}_1(D)$, where D is a finite-dimensional non-commutative division algebra over F . Then G admits a (finite-dimensional) irreducible smooth representation of dimension greater than 1, which becomes reducible in $\mathcal{O}^{\mathfrak{g}}$.

Part (ii) may require an extension of scalars even when \underline{G} is a torus, cf. Remark 2.21.

We cannot drop the condition that $\underline{G}^{\text{der}}$ is simply connected. For parts (i) and (ii), suppose $\underline{G} = \text{PGL}_3$, $F = \mathbb{Q}_p$, $p \equiv 1 \pmod{3}$. Then we have $L(2/3, -1/3, -1/3) \in \mathcal{O}^{\mathfrak{g}}$, and we claim that it does not lift to \mathcal{O}^G . If there was a lift in \mathcal{O}^G , then we can inflate it to $\mathcal{O}^{\text{GL}_3}$, so by part (iii) it is of the form $L(1, 0, 0) \otimes (\psi \circ \det)$, where ψ is a locally analytic character of \mathbb{Q}_p^\times . But by considering the 3-torsion subgroup of \mathbb{Q}_p^\times it is easy to see that such a lift does not exist, as we cannot solve $\psi(x)^3 = x^{-1}$

($x \in \mathbb{Q}_p^\times$). A similar example with $\underline{G} = \mathrm{PGL}_2$ can be found in [OS14b, Example 2.4]. For part (iii), suppose $\underline{G} = \mathrm{PGL}_3$, $F = \mathbb{Q}_p$, $p \equiv 2 \pmod{3}$, and $p^{1/3} \in C$. Then there exists a continuous (hence locally \mathbb{Q}_p -analytic) character $\psi: \mathbb{Q}_p^\times \rightarrow C^\times$ such that $\psi(x)^3 = x^{-1}$. Let V be the standard representation of $\mathrm{GL}_3(\mathbb{Q}_p)$. Then $V \otimes (\psi \circ \det) \in \mathcal{O}^G$ but is not an algebraic representation up to twist. (The last example is related to [STP01, §3 Example].)

Proof. (i) Take any $M \in \mathcal{O}^{\mathfrak{g}}$ simple. Then $M \cong L(\lambda)$ for some $\lambda \in (\mathfrak{t}')^*$ with $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \tilde{\Phi}^+$. As \underline{G} is semisimple simply-connected, $\lambda \in X^*(\underline{T}')$ is dominant and so M lifts to the algebraic representation $L(\lambda)$ of \underline{G}_C ; by restriction to G we get our desired object of $\mathcal{O}_{\mathrm{alg}}^G$.

If moreover all simple factors of \underline{G} are isotropic, then by Kneser–Tits we know that G is generated by the unipotent radicals U and \bar{U} . If $M_1, M_2 \in \mathcal{O}^G$ and $f: M_1 \rightarrow M_2$ is $U(\mathfrak{g}_C)$ -linear, then by integrating the action over unipotents we deduce that f is U -linear and \bar{U} -linear, hence G -linear. Therefore the forgetful functor $\mathcal{O}^G \rightarrow \mathcal{O}^{\mathfrak{g}}$ is fully faithful. By the previous paragraph the composite $\mathcal{O}_{\mathrm{alg}}^G \rightarrow \mathcal{O}^G \rightarrow \mathcal{O}^{\mathfrak{g}}$ of fully faithful embeddings is an essential surjection (as $\mathcal{O}^{\mathfrak{g}}$ is semisimple), hence we get equivalences $\mathcal{O}_{\mathrm{alg}}^G = \mathcal{O}^G = \mathcal{O}^{\mathfrak{g}}$.

(ii) First we consider the case $P = G$ and take $W \in \mathcal{O}^{\mathfrak{g}}$ (absolutely) simple. By the above we can lift $W|_{\mathfrak{g}_C^{\mathrm{der}}}$ to an algebraic representation of G_C^{der} , which extends to an algebraic representation of G_C (as $\underline{T}'^{\mathrm{der}}$ is a direct factor of \underline{T}'). So there exists an algebraic G -representation M that agrees with W on $\mathfrak{g}_C^{\mathrm{der}}$. The space $\lambda := \mathrm{Hom}_{\mathfrak{g}_C^{\mathrm{der}}}(M, W)$ has a natural action of $\mathfrak{g}_C/\mathfrak{g}_C^{\mathrm{der}}$ and, since M, W are absolutely simple $\mathfrak{g}_C^{\mathrm{der}}$ -modules, λ is 1-dimensional. Hence $M \otimes \lambda \rightarrow W$ is an isomorphism in $\mathcal{O}^{\mathfrak{g}}$. By Lemma 2.20 (extending scalars if necessary) we can lift $\lambda: \mathfrak{g}_C/\mathfrak{g}_C^{\mathrm{der}} \rightarrow C$ to a locally analytic character $G/G^{\mathrm{der}} \rightarrow C^\times$, and this implies the claim.

Now suppose P is arbitrary standard parabolic subgroup and $W \in \mathcal{O}^P$ simple. Then $W^{\mathrm{nc}} \in \mathcal{O}^I$ is simple and we can lift it to an object M' of \mathcal{O}^L by the previous paragraph (after a scalar extension). Then the unique simple quotient $\underline{L}(M')$ of $\underline{M}(M')$ in \mathcal{O}^P lifts W by Lemma 2.11.

If $M_1, M_2 \in \mathcal{O}^P$ are lifts of the same simple object of \mathcal{O}^P , then $M_1^N|_{\mathfrak{t}_C} \cong M_2^N|_{\mathfrak{t}_C}$ and we let $\eta := \mathrm{Hom}_{\mathfrak{t}_C}(M_1^N, M_2^N)$, a smooth character of L . Then $M_1^N \otimes \eta \cong M_2^N$ in \mathcal{O}^L and hence $M_1 \otimes \eta \cong M_2$ in \mathcal{O}^P by Lemma 2.11.

(iii) In the proof of part (ii) we saw that $M|_{\mathfrak{g}_C}$ admits a lift of the form $M' = M_{\mathrm{alg}} \otimes \psi'$ in \mathcal{O}^G , where M_{alg} is algebraic and ψ' a character of G/G^{der} . Then $\eta := \mathrm{Hom}_{\mathfrak{g}_C}(M', M)$ is a 1-dimensional smooth representation of G and $M \cong M' \otimes \eta \cong M_{\mathrm{alg}} \otimes \psi' \eta$. \square

Proposition 2.24.

- (i) If $M_0 \in \mathcal{O}^P$ is \mathfrak{g}_C -simple and τ is finite-dimensional irreducible (resp. absolutely irreducible) smooth L -representation, then $M_0 \otimes \tau$ is a simple (resp. absolutely simple) object of \mathcal{O}^P .
- (ii) The decomposition $M_0 \otimes \tau$ in (i) is unique up to a smooth character of L .
- (iii) If $\underline{G}^{\mathrm{der}}$ is simply connected, then any absolutely simple object of \mathcal{O}^P can be decomposed as in (i), after perhaps replacing C by a finite extension.

Proof. (i) Consider $M = M_0 \otimes \tau$, where $M_0 \in \mathcal{O}^P$ is \mathfrak{g}_C -simple and τ is a finite-dimensional irreducible smooth L -representation. Then M lies in \mathcal{O}^P clearly. By restricting to \mathfrak{g}_C we see that any subobject is of the form $M_0 \otimes V'$ for some L -stable subspace $V' \subset \tau$, hence M is simple. If moreover τ is absolutely irreducible, then M is absolutely simple.

(ii) Suppose $M_1 \otimes \tau_1 \cong M_2 \otimes \tau_2$ in \mathcal{O}^P with M_i, τ_i as in the statement of part (i). By taking N -invariants we get $M_1^N \otimes \tau_1 \cong M_2^N \otimes \tau_2$ in \mathcal{O}^L , where $M_i^N \in \mathcal{O}^L$ are \mathfrak{l}_C -simple.

Restricting to \mathfrak{l}_C we deduce $M_1^N|_{\mathfrak{l}_C} \cong M_2^N|_{\mathfrak{l}_C}$. Then $\eta := \text{Hom}_{\mathfrak{l}_C}(M_1^N, M_2^N)$ is a smooth 1-dimensional representation of L and $M_1^N \otimes \eta \cong M_2^N$. By replacing (M_1, τ_1) by $(M_1 \otimes \eta, \tau_1 \otimes \eta^{-1})$ we may assume $\eta = 1$. From $\text{Hom}_{\mathfrak{l}_C}(M_1^N, M_1^N \otimes \tau_i) \cong \tau_i$ we then deduce $\tau_1 \cong \tau_2$. Finally we deduce $M_1 \cong M_2$ from $\underline{M}(M_1^N) \cong \underline{M}(M_2^N)$ and Lemma 2.11.

(iii) We first treat the case where $P = G$, so suppose $M \in \mathcal{O}^G$ absolutely simple. Pick W_0 a simple subobject of M in $\mathcal{O}^{\mathfrak{q}}$. Extending scalars if necessary, we can find $M_0 \in \mathcal{O}^G$ lifting W_0 by Lemma 2.22 (using that $\underline{G}^{\text{der}}$ is simply connected). We now have absolutely irreducible G -representations M_0, M such that

$$\tau := \text{Hom}_{\mathfrak{g}_C}(M_0, M) \neq 0.$$

Note that G acts smoothly on τ and that we get a nonzero G -linear map $M_0 \otimes \tau \rightarrow M$. This is an isomorphism, as the both sides are simple. It follows that τ is absolutely irreducible.

Now in general, suppose that $M \in \mathcal{O}^P$. Note that $M^N \in \mathcal{O}^L$ is absolutely simple by Lemma 2.11. Write $M^N = M_0 \otimes \tau$ by the previous paragraph (after perhaps extending scalars), where $M_0 \in \mathcal{O}^L$ is \mathfrak{l}_C -simple and τ is an absolutely irreducible smooth L -representation. Then $M \cong \underline{L}(M^N) \cong \underline{L}(M_0) \otimes \tau$ by Lemma 2.13 and $\underline{L}(M_0) \in \mathcal{O}^P$ is \mathfrak{g}_C -simple. \square

Corollary 2.25. *Suppose that $P = LN$ and that all simple factors of the adjoint group $\underline{L}^{\text{ad}} = \underline{L}/\underline{Z}_{\underline{L}}$ are isotropic. Then every absolutely simple object of \mathcal{O}^P is \mathfrak{g}_C -simple, i.e. a decomposition as in Proposition 2.24(i) exists (with $\tau = 1$).*

Proof. Suppose that $M \in \mathcal{O}^P$ is absolutely simple. If $\underline{G}^{\text{der}}$ is simply connected, then we can write $M \cong M_0 \otimes \tau$ as in Proposition 2.24. As $\underline{L}^{\text{der}}$ is simply connected and all simple factors of $\underline{L}^{\text{der}}$ are isotropic, by Kneser–Tits we know that $L' = L^{\text{der}}$. Hence the finite-dimensional smooth representation τ is trivial on L^{der} by Lemma 2.4. Since L/L^{der} is abelian, we deduce that τ is 1-dimensional and hence M is \mathfrak{g}_C -simple.

For general \underline{G} , let $\tilde{\underline{G}} \rightarrow \underline{G}$ be a z -extension, so $\tilde{\underline{G}}^{\text{der}}$ is simply connected, $\tilde{\underline{G}} \rightarrow \underline{G}$ on F -points, and $\tilde{\mathfrak{g}}_C \rightarrow \mathfrak{g}_C$ for the Lie algebras. Then the inflation \tilde{M} of M becomes an absolutely simple object of $\mathcal{O}^{\tilde{P}}$, where \tilde{P} is the pre-image of P in $\tilde{\underline{G}}$. By the previous paragraph, \tilde{M} is $\tilde{\mathfrak{g}}_C$ -simple, so M is \mathfrak{g}_C -simple. \square

Proposition 2.26. *Suppose that $\underline{G}^{\text{der}}$ is simply connected.*

- (i) *If M is a \mathfrak{g}_C -simple object in $\mathcal{O}^B \cap \mathcal{O}^{\mathfrak{p}}$, then, perhaps after replacing C by a finite extension, there is a smooth character η of Z such that $M \otimes \eta$ lies in \mathcal{O}^P . Moreover, η is unique up to a smooth character of L .*
- (ii) *If M is a \mathfrak{g}_C -simple object of \mathcal{O}^P , then, perhaps after replacing C by a finite extension, there is a smooth character η of L such that $M \otimes \eta$ is equimaximal. Moreover, η is unique up to a smooth character of L_Q , where Q is the maximal parabolic such that $M \in \mathcal{O}^{\mathfrak{q}}$.*

Remark 2.27. We cannot drop the condition that $\underline{G}^{\text{der}}$ is simply connected in part (i) (see the example with $G = \text{PGL}_3(F)$, $p \equiv 1 \pmod{3}$ in Remark 2.23).

Proof. (i) Take M a \mathfrak{g}_C -simple object in $\mathcal{O}^B \cap \mathcal{O}^{\mathfrak{p}}$. By Lemma 2.22(ii) there exists $W \in \mathcal{O}^P$ such that $W|_{\mathfrak{g}_C} \cong M|_{\mathfrak{g}_C}$ and moreover there exists a smooth character η of Z such that $W \cong M \otimes \eta$ in \mathcal{O}^B . The uniqueness follows from the uniqueness of Lemma 2.22(ii).

(ii) By part (i) there exists a smooth character η of Z such that $M \otimes \eta \in \mathcal{O}^Q$, i.e. $M \otimes \eta$ is equimaximal. By the uniqueness part of (i) applied to $M, M \otimes \eta \in \mathcal{O}^P$, we see that η is in fact a smooth character of L . The uniqueness assertion follows by the same reasoning. \square

We can now prove Corollary 2.18.

Proof of Corollary 2.18. The strong admissibility follows from the proof of [OS15, Lemma 2.4(ii)]. (By Lemma 2.4 any irreducible finite-dimensional locally analytic representation of P is trivial on N .)

For the finite length claim, we first make a reduction to the case where $\underline{G}^{\text{der}}$ is simply connected. We take a z -extension $\tilde{G} \rightarrow \underline{G}$, so \tilde{G}^{der} is simply connected and $\tilde{G} \rightarrow G$ on F -points. By pullback to \tilde{G} we obtain $\tilde{P} = \tilde{L}\tilde{N}$, and inflation gives $\tilde{M} \in \mathcal{O}^{\tilde{P}}$ and $\tilde{\pi}$ finite-length smooth of \tilde{L} . The construction of Orlik–Strauch is compatible with pullback, i.e. the inflation of $\mathcal{F}_P^G(M, \pi)$ to \tilde{G} is naturally isomorphic to $\mathcal{F}_{\tilde{P}}^{\tilde{G}}(\tilde{M}, \tilde{\pi})$. Thus if $\mathcal{F}_{\tilde{P}}^{\tilde{G}}(\tilde{M}, \tilde{\pi})$ is topologically of finite length, so is $\mathcal{F}_P^G(M, \pi)$.

Suppose $\underline{G}^{\text{der}}$ is simply connected. By exactness of \mathcal{F}_P^G , we may assume that M is simple. It suffices to prove the result after a finite scalar extension, and we allow such extensions in the proof without further comment. In particular, we may assume that M is absolutely simple. By Proposition 2.24(iii) we can write $M = M_0 \otimes \tau$ with M_0 \mathfrak{g}_C -simple and τ a finite-dimensional smooth L -representation. Using Proposition 2.26(ii) we may moreover twist by a smooth character of L and assume that M_0 is equimaximal. By construction, $\mathcal{F}_P^G(M, \pi) \cong \mathcal{F}_P^G(M_0, \pi \otimes \tau)$. Moreover, $\pi \otimes \tau$ is of finite length, since it is admissible and finitely generated. Hence we may assume that M is \mathfrak{g}_C -simple and equimaximal. By exactness of \mathcal{F}_P^G we can also assume that π is irreducible. Then $\mathcal{F}_P^G(M, \pi)$ is irreducible by Theorem 2.16. \square

Lemma 2.28. *Suppose that $W \in \mathcal{O}^L$ is \mathfrak{l}_C -simple. Then every subobject of $\underline{M}(W)$ in \mathcal{O}^P is P -stable, i.e. lies in \mathcal{O}^P . In particular, every Jordan–Hölder factor of $\underline{M}(W)$ in \mathcal{O}^P is \mathfrak{g}_C -simple.*

Proof. Suppose first that $\underline{G}^{\text{der}}$ is simply connected. It suffices to check this after a finite scalar extension. So by Lemma 2.22 we may write $W = W_{\text{alg}} \otimes \psi$ with W_{alg} algebraic and ψ a locally analytic character of L . By integration, every subobject of $\underline{M}(W)$ in \mathcal{O}^P is N -stable. As an L -representation we have $\underline{M}(W) = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W \cong U(\mathfrak{u}_{P,C}^-) \otimes W_{\text{alg}} \otimes \psi$, and by twisting it suffices to check L -stability when $\psi = 1$. As $U(\mathfrak{u}_{P,C}^-) \otimes W_{\text{alg}}$ is a (locally finite) algebraic representation of L , it is clear that any \mathfrak{l}_C -submodule is L -stable. For general \underline{G} , take a z -extension $\varphi : \tilde{G} \rightarrow \underline{G}$, so \tilde{G}^{der} is simply connected and $\tilde{G} \rightarrow G$ on F -points. Let $\tilde{P} := \varphi^{-1}(P)$, $\tilde{L} := \varphi^{-1}(L)$ and let $\tilde{W} \in \mathcal{O}^{\tilde{L}}$ be obtained by inflation (cf. Remark 2.9). Then $\underline{M}_{\tilde{G}}(\tilde{W}) \in \mathcal{O}^{\tilde{P}}$ is obtained from $\underline{M}_G(W) \in \mathcal{O}^P$ by inflation, and the claim follows from the previous case. \square

Lemma 2.29. *Suppose that $W \in \mathcal{O}^L$ is absolutely simple and that $Q = L_Q N_Q$ is any parabolic subgroup containing P .*

- (i) *Suppose that $W \cong W_0 \otimes \tau$ in \mathcal{O}^L , where W_0 is \mathfrak{l}_C -simple and τ is smooth. Then $\underline{L}(W) \in \mathcal{O}^Q$ if and only if $\underline{L}(W_0) \in \mathcal{O}^Q$ and τ extends to a smooth representation of L_Q .*
- (ii) *Suppose that $\underline{G}^{\text{der}}$ is simply connected. Then $\underline{L}(W) \in \mathcal{O}^Q$ if and only if (after perhaps replacing C by a finite extension) $W \cong W_{\text{alg}} \otimes \psi|_L \otimes \tilde{\tau}|_L$ for some algebraic representation W_{alg} of L , a locally analytic character ψ of*

L_Q , and a smooth representation $\tilde{\tau}$ of L_Q such that moreover $L(W_{\text{alg}}) \in \mathcal{O}^{\mathfrak{q}}$.

In part (ii) we could alternatively demand that $W \cong W_{\text{alg}} \otimes \psi|_L \otimes \tau$, where τ is a smooth representation L that is trivial on $L \cap L'_Q$ (after perhaps replacing C by a finite extension). (If $\tilde{\tau}$ is as above, then it is trivial on L'_Q by Lemma 2.4. Conversely, τ is a representation of $L/(L \cap L'_Q) \cong L_Q/L'_Q$, so extends to L_Q .) Also note that the property $L(W_{\text{alg}}) \in \mathcal{O}^{\mathfrak{q}}$ is equivalent to the lowest weight of W_{alg} in $X^*(\underline{T}')$ being antidominant relative to the Levi \underline{L}_Q .

Proof. (i) Suppose that $\underline{L}(W) \in \mathcal{O}^Q$ and $W \cong W_0 \otimes \tau$ as in the statement of the lemma. Then $\underline{L}(W) \cong \underline{L}(W_0) \otimes \tau$ in \mathcal{O}^P by Lemma 2.13. Moreover $\underline{L}(W)^{N_Q} \in \mathcal{O}^{L_Q}$ is finite-dimensional and by Lemma 2.4 we have $\underline{L}(W)^{N_Q} \cong \underline{L}(W_0)^{N_Q} \otimes \tau$. Hence $\underline{L}(W_0)^{N_Q}$ is finite-dimensional, hence lies in \mathcal{O}^{L_Q} , so $\underline{L}(W_0) \in \mathcal{O}^Q$ as it is a quotient of $\underline{M}(\underline{L}(W_0)^{N_Q}) \in \mathcal{O}^Q$. Let $\tilde{\tau} := \text{Hom}_{\mathfrak{g}_C}(\underline{L}(W_0), \underline{L}(W))$. Then $\tilde{\tau}$ has a smooth action of L_Q and $\tilde{\tau}|_L \cong \tau$ by the isomorphism above.

Conversely, let $\tilde{\tau}$ be the (unique) smooth extension of τ to L_Q . Then $\underline{L}(W) \cong \underline{L}(W_0) \otimes \tilde{\tau} \in \mathcal{O}^Q$ because they are isomorphic in \mathcal{O}^P .

(ii) If $\underline{G}^{\text{der}}$ is simply connected, then we can always decompose $W \cong W_0 \otimes \tau$ as in (i) by Proposition 2.24 (after perhaps extending scalars). If $\underline{L}(W) \in \mathcal{O}^Q$, then by (i) we deduce that τ extends to a smooth representation $\tilde{\tau}$ of L_Q and that $\underline{L}(W_0) \in \mathcal{O}^Q$. By Lemma 2.22(iii) we can write $\underline{L}(W_0)^{N_Q} \cong M_{\text{alg}} \otimes \psi$ in \mathcal{O}^{L_Q} with M_{alg} algebraic and ψ a locally analytic character of L_Q (after perhaps extending scalars). Taking $N \cap L_Q$ -invariants and using Lemma 2.4 we get that $W_0 \cong M_{\text{alg}}^{N \cap L_Q} \otimes \psi|_L$ with $M_{\text{alg}}^{N \cap L_Q}$ algebraic.

Conversely, if $W \cong W_{\text{alg}} \otimes \psi|_L \otimes \tilde{\tau}|_L$ as in the statement of the lemma, then $\underline{L}(W_{\text{alg}})$ is equimaximal as W_{alg} is algebraic (Lemma 2.19), so $\underline{L}(W_{\text{alg}}) \in \mathcal{O}^Q$. Let $W_1 := W_{\text{alg}} \otimes \psi|_L$, which is an \mathfrak{l}_C -simple object of \mathcal{O}^L , and $\underline{L}(W_1) \cong \underline{L}(W_{\text{alg}}) \otimes \psi|_L$ in \mathcal{O}^P by Lemmas 2.13, 2.4 and shows that $\underline{L}(W_1) \cong \underline{L}(W_{\text{alg}}) \otimes \psi \in \mathcal{O}^Q$. Then $W \cong W_1 \otimes \tilde{\tau}|_L$ lies in \mathcal{O}^Q by (i). \square

2.5. The socle of locally analytic parabolic induction. The following proposition generalizes [OS14a, Theorem 3.5] and [Bre16, Proposition 2.4], which assumed $M \in \mathcal{O}_{\text{alg}}^P$ and \underline{G} split.

Proposition 2.30. *Suppose that $P = LN$ and that $M \in \mathcal{O}^P$ is \mathfrak{g}_C -simple and equimaximal with maximal parabolic P . Let π be an admissible smooth representation of L . Then*

$$(2.2) \quad H^0(N, \mathcal{F}_P^G(M, \pi)') \cong M^N \otimes \pi'$$

as representations of L (on nuclear Fréchet spaces).

Proof. The proof follows the same lines as [OS14a, Theorem 3.5] and [Bre16, Proposition 2.4]. We let $\overline{P} = L\overline{N}$ denote the opposite parabolic subgroup and $\overline{\mathfrak{n}}$ the Lie algebra of \overline{N} .

We fix some notation. Recall that we fixed a special parahoric subgroup $K \subset G$. Let $P_0 := P \cap K$. For X a locally F -analytic manifold, let $D(X)$ be the locally convex space of locally F -analytic distributions on X with coefficients in C . If H is a locally F -analytic group, then $D(H)$ is a (separately continuous) locally convex algebra. For $h \in H$, we have $\delta_h \in D(H)$ defined by $f \mapsto f(h)$. If H is a closed subgroup of G , let $D(\mathfrak{g}, H)$ be the subalgebra of $D(G)$ generated by $U(\mathfrak{g}_C)$ and $D(H)$. Let Φ be the root system for $(\underline{G}, \underline{S})$, Φ^+ the set of roots in \underline{B} , Δ the set of simple roots and W the Weyl group.

First assume $\pi = 1$. Assume $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$ and r sufficiently close to 1 (where $\kappa \in \{1, 2\}$ is as in §A). Let $I \subset \Delta$ be the set of simple roots corresponding to P and W_I the subset of W generated by reflections for elements in I . For each $w \in W$ we fix a representative $\dot{w} \in K$. Let \mathcal{I} be an Iwahori subgroup fixing a facet of the apartment of \underline{S} having vertex x_0 . Then

$$\mathcal{F}_P^G(M)' \cong \bigoplus_{w \in W^I} \delta_{\dot{w}} D(\dot{w}^{-1} \mathcal{I} \dot{w} \cdot P_0) \otimes_{D(\mathfrak{g}, P_0)} M,$$

where W^I denotes the Kostant representatives of W/W_I . Fix w and let $\hat{H} := \dot{w}^{-1} \mathcal{I} \dot{w}$, $\mathcal{M} := D(\hat{H} P_0) \otimes_{D(\mathfrak{g}, P_0)} M$ (coadmissible). We now use the notation and definitions of Appendix A, in particular defining G_0 and an open normal L -uniform subgroup $H \triangleleft G_0$ that has an Iwahori decomposition $H = H^- H^+$ with respect to $\bar{N} \times P$. We may assume that H is contained in \hat{H} . The group \hat{H} is used to define norms q_r (cf. [OS10, 2.2.6]) on $D(\hat{H})$ for any compact subgroup \tilde{H} of G that contains H (and likewise H^\pm is used to define norms on $D(\tilde{H}^\pm)$ for any compact subgroup \tilde{H}^\pm of \bar{N} , resp. P , that contains H^\pm). We also have an Iwahori decomposition $\hat{H} = \hat{H}^- \hat{H}^+$ with respect to $\bar{N} \times P$. Let $D_r(H)$ be the completion of $D(H)$ with respect to q_r . Let $U_r(\mathfrak{g})$ denote the closure of $U(\mathfrak{g}_C)$ in $D_r(H)$ (or equivalently in $D_r(\hat{H})$) and $D_r(\mathfrak{g}, P_0)$ the subring of $D_r(K)$ generated by $U_r(\mathfrak{g})$ and $D_r(P_0)$. Let $\mathcal{M}_r := D_r(\hat{H} P_0) \otimes_{D(\hat{H} P_0)} \mathcal{M} \cong D_r(\hat{H}) \otimes_{D(\mathfrak{g}, \hat{H} \cap P_0)} M$ (so $\mathcal{M} \cong \varprojlim_r \mathcal{M}_r$ by coadmissibility) and $\mathfrak{m}_r := U_r(\mathfrak{g}) M \subset \mathcal{M}_r$. As in the proof of [OS14b, Theorem 4.5] the module \mathfrak{m}_r is $D_r(\mathfrak{g}, P_0)$ -stable, $D_r(\hat{H} P_0) = \bigoplus_{\hat{H} P_0 / H^m P_0} \delta_g D_r(\mathfrak{g}, P_0)$, $\mathfrak{m}_r \cong D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} M$, and $\mathcal{M}_r \cong \bigoplus_{\hat{H}^- / H^{-,m}} \delta_u \mathfrak{m}_r$, where H^m (resp. $H^{-,m}$) is the $(m+1)$ -st term in the lower p -series of H (resp. H^-). By Lemma A.10 we get $\mathfrak{m}_r \cong U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g}_C)} M$ and Lemma A.11 applies to $M \subset \mathfrak{m}_r$ (by the beginning of the proof of Theorem A.13).

If $w \in W^I \setminus \{1\}$, then there exists a reduced root $\beta \in \Phi^+ \setminus \Phi_I^+$ such that $w^{-1} \beta \in \Phi^- \setminus \Phi_I^-$. (For any $w \notin W_I$ we can write $w = w_1 w' w_2$ with $w_i \in W_I$ and $w' \neq 1$ the Kostant representative for the double coset. Then there exists a reduced $\beta > 0$ such that $w'^{-1} \beta < 0$.) If moreover $H^0(\mathfrak{n}_C, \delta_{\dot{w}} \mathcal{M}_r) \neq 0$, then $H^0(\mathfrak{n}_C, \delta_{\dot{w}} \delta_u \mathfrak{m}_r) \neq 0$ for some $u \in \hat{H}^-$, so $\text{Ad}(u^{-1})y$ fails to act injectively on \mathfrak{m}_r for any $y \in \mathfrak{g}_{(w^{-1}\beta), C} \subset \text{Ad}(w^{-1})\mathfrak{n}_C \cap \bar{\mathfrak{n}}_C$. Arguing as in the proof of Theorem A.13, by equimaximality, we deduce that $-w^{-1} \beta \in \Phi_I^+$, contradiction.

Now suppose $w = 1$ and $H^0(\mathfrak{n}_C, \delta_u \mathfrak{m}_r) \neq 0$ for some $u \in \hat{H}^-$. Note that \mathfrak{m}_r is simple as $U_r(\mathfrak{g})$ -module by Theorem A.13. As in Step 2 of the proof of [OS14b, Theorem 4.7] we can embed \mathfrak{m}_r into the formal completion $\widehat{M} = \prod_{\lambda \in \mathfrak{a}_P^*} M_\lambda$, where \mathfrak{a}_P denotes the Lie algebra of the maximal split torus of the center of \underline{L} , where each λ -weight space M_λ is finite-dimensional. The action of $\bar{\mathfrak{n}}$ exponentiates to a locally analytic action of \bar{N} and we have $\bar{n} \circ X \circ \bar{n}^{-1} = \text{Ad}(\bar{n})(X)$ on \widehat{M} for all $\bar{n} \in \bar{N}$, $X \in \mathfrak{g}_C$. (Use, for example, that $\log(\text{Ad}(\bar{n})) = \text{ad}(\log(\bar{n})) \in \text{GL}(\mathfrak{g}_C)$ for all $\bar{n} \in \bar{N}$.) By assumption, $0 \neq H^0(\text{Ad}(u^{-1})\mathfrak{n}_C, \mathfrak{m}_r) = \mathfrak{m}_r \cap u^{-1} \widehat{M}^{\mathfrak{n}_C} = \mathfrak{m}_r \cap u^{-1} M^{\mathfrak{n}_C}$ inside \widehat{M} , so $M^{\mathfrak{n}_C} \hookrightarrow u \mathfrak{m}_r$ as \mathfrak{p}_C -modules, hence we get a surjective map $U_r(\mathfrak{g}) \otimes_{U(\mathfrak{p}_C)} M^{\mathfrak{n}_C} \twoheadrightarrow \delta_u \star \mathfrak{m}_r$ of $U_r(\mathfrak{g})$ -modules. (Here, $\delta_u \star \mathfrak{m}_r$ denotes the space \mathfrak{m}_r equipped with the action of $U_r(\mathfrak{g})$ twisted by δ_u .) By Lemma A.11 the left-hand side has \mathfrak{m}_r as unique simple quotient, so we get $\mathfrak{m}_r \xrightarrow{\sim} \delta_u \star \mathfrak{m}_r$, hence $u \in H^{-,m}$ by Theorem A.13.

Therefore, $H^0(\mathfrak{n}_C, \mathcal{M}_r) = H^0(\mathfrak{n}_C, \mathfrak{m}_r) = M^{\mathfrak{n}_C}$, giving an isomorphism $M^{\mathfrak{n}_C} \cong H^0(\mathfrak{n}_C, \mathcal{F}_P^G(M)')$. More precisely this is the image of the map $i : M^N = M^{\mathfrak{n}_C} \hookrightarrow M \rightarrow D(G) \otimes_{D(\mathfrak{g}, P)} M \cong \mathcal{F}_P^G(M)'$ sending x to $\delta_1 \otimes x$. Using a choice of locally analytic section $s : G/P \rightarrow G$ of the projection $G \twoheadrightarrow G/P$ we obtain \mathfrak{g}_C -linear isomorphisms $\mathcal{F}_P^G(M, \pi) \cong \mathcal{F}_P^G(M) \widehat{\otimes} \pi$ and $\mathcal{F}_P^G(M, \pi)' \cong \mathcal{F}_P^G(M)' \widehat{\otimes} \pi'$. We may take

s such that $s(\bar{1}) = 1$. As in [OS14a, Theorem 3.5] we find that $H^0(\mathfrak{n}_C, \mathcal{F}_P^G(M, \pi)')$ is the image of the map

$$(2.3) \quad i \otimes 1: M^N \otimes \pi' \hookrightarrow \mathcal{F}_P^G(M)' \widehat{\otimes} \pi' \cong \mathcal{F}_P^G(M, \pi)'$$

whose strong dual is computed to be $\mathcal{F}_P^G(M, \pi) \hookrightarrow (\text{Ind}_P^G(M^N)' \otimes \pi)^{\text{an}} \twoheadrightarrow (M^N)' \otimes \pi$, where the second map is $f \mapsto f(1)$ (using $s(\bar{1}) = 1$), which is P -linear (this is not clear a priori). Taking N -invariants in (2.3) we finally obtain the L -linear isomorphism (2.2). \square

The following corollary generalizes [Bre16, Corollary 2.5], which assumed $M \in \mathcal{O}_{\text{alg}}^P$ and \underline{G} split.

Corollary 2.31. *Suppose that $P = LN$ and that $M \in \mathcal{O}^P$ is \mathfrak{g}_C -simple and equimaximal with maximal parabolic $Q = L_Q N_Q$. Let π be an (admissible) smooth representation of L of finite length. Then*

$$\begin{aligned} \text{soc}_G \mathcal{F}_P^G(M, \pi) &= \mathcal{F}_Q^G(M, \text{soc}_{L_Q}(\text{Ind}_{P \cap L_Q}^{L_Q} \pi)^{\text{sm}}) \\ &= \text{soc}_G((\text{Ind}_P^G(M^N)' \otimes \pi)^{\text{an}}). \end{aligned}$$

Remark 2.32. We note that any finite-length smooth C -representation of G is admissible. For the proof, we can use a classical argument as follows. We may assume that the representation is irreducible. Let π be an irreducible smooth representation. Then there exists a parabolic subgroup $P = LU$ and an irreducible cuspidal representation σ such that $\pi \hookrightarrow (\text{Ind}_P^G \sigma)^{\text{sm}}$ [Vig96, II.2.4]. Since $(\text{Ind}_P^G -)^{\text{sm}}$ preserves admissible representations, it is sufficient to prove that σ is admissible. Hence we may assume that π is cuspidal.

Since G is σ -compact, $\dim_C \pi$ is countable. Hence $\dim_C \text{End}_G(\pi)$ is countable. Since C is uncountable, the division algebra $\text{End}_G(\pi)$ cannot contain a field of rational functions. Therefore for any $z \in Z_G$, the image of z in $\text{End}_G(\pi)$ is algebraic over C . Since Z_G is topologically finitely generated, after tensoring with a finite extension of C and taking an irreducible subquotient, we may assume that π has a central character. Here note that a representation π is admissible (resp. finite length) if and only if $\pi \otimes_C C'$ is admissible (resp. finite length) for a finite extension C' . By [Vig96, II.2.7], π is Z_G -compact. Let K' be a compact open subgroup of G and let $e_{K'}: V \rightarrow V^{K'}$ be the K' -equivariant projection. Fix a nonzero vector $v \in \pi$ and set $D := \{g \in G \mid e_{K'} gv \neq 0\}$. Then D/Z_G is compact by [Vig96, I.7.3]. Therefore $K' \backslash D/Z_G$ is finite. Since π is irreducible, $\pi = \sum_{g \in G} Cgv$. Hence $\pi^{K'} = \sum_{g \in G} C e_{K'} gv = \sum_{g \in K' \backslash D/Z_G} \sum_{z \in Z_G} C e_{K'} g z v$. The representation π has a central character, hence $C e_{K'} g z v = C e_{K'} gv$. Therefore $\pi^{K'} = \sum_{g \in K' \backslash D/Z_G} C e_{K'} gv$ is finite-dimensional.

Proof of Corollary 2.31. The proof proceeds as in [Bre16], using Proposition 2.30 instead of [Bre16, Proposition 2.4]. We first make a reduction to the case where $\underline{G}^{\text{der}}$ is simply connected. We take a z -extension $1 \rightarrow \underline{T} \rightarrow \underline{\tilde{G}} \rightarrow \underline{G} \rightarrow 1$, where $\underline{\tilde{G}}^{\text{der}}$ is simply connected and \underline{T} is a central induced torus, so $1 \rightarrow T \rightarrow \tilde{G} \rightarrow G \rightarrow 1$. By pullback to \tilde{G} we obtain $\tilde{P} = \tilde{L}\tilde{N}$ and $\tilde{Q} = \tilde{L}_Q \tilde{N}_Q$, and by inflation we obtain $\tilde{M} \in \mathcal{O}^{\tilde{P}}$ and $\tilde{\pi}$ finite-length smooth of \tilde{L} . Note that \tilde{M} is $\tilde{\mathfrak{g}}_C$ -simple and equimaximal with maximal parabolic \tilde{Q} . Moreover, the construction of Orlik–Strauch is compatible with pullback, i.e. the inflation of $\mathcal{F}_P^G(M, \pi)$ to \tilde{G} is naturally isomorphic to $\mathcal{F}_{\tilde{P}}^{\tilde{G}}(\tilde{M}, \tilde{\pi})$ and likewise for $\mathcal{F}_P^G(\underline{M}(M^N), \pi) = (\text{Ind}_P^G(M^N)' \otimes \pi)^{\text{an}}$. This completes the reduction.

It is clear that the functor \mathcal{F}_P^G commutes with finite extensions of scalars. Also, the functor soc_G (resp. soc_{L_Q}) commutes with finite extensions of scalars on the

category of admissible locally analytic representations of G (resp. L_Q). (This follows, upon dualizing, from the statement that for a module M over a C -algebra D we have $\text{soc}_D(M) \otimes_C C' \cong \text{soc}_{D \otimes_C C'}(M \otimes_C C')$, for any finite separable extension C'/C . On the one hand, M is semisimple over D if and only if $M \otimes_C C'$ is semisimple over $D \otimes_C C'$, cf. [Bou12, §12.7]. On the other hand, $\text{soc}_{D \otimes_C C'}(M \otimes_C C')$ is defined over C , by Galois descent.) Therefore it is enough to prove the result after a finite extension of scalars. So by Proposition 2.26 we may assume that each simple constituent of $\underline{M}(M^N)$ in \mathcal{O}^P is equimaximal up to twist by a smooth character of L . (By Lemma 2.28 all such constituents are \mathfrak{g}_C -simple.)

By the equimaximality assumption the irreducible constituents of $\mathcal{F}_P^G(M, \pi)$ are precisely the $\mathcal{F}_Q^G(M, \pi_Q)$, where π_Q is an irreducible constituent of $(\text{Ind}_{P \cap L_Q}^{L_Q} \pi)^{\text{sm}}$. If $\mathcal{F}_Q^G(M, \pi_Q)$ injects into $\mathcal{F}_P^G(M, \pi) \cong \mathcal{F}_Q^G(M, I)$ where $I := (\text{Ind}_{P \cap L_Q}^{L_Q} \pi)^{\text{sm}}$ and π_Q an irreducible constituent of I , then from Proposition 2.30 we get an L_Q -linear map $M^{N_Q} \otimes I' \rightarrow M^{N_Q} \otimes \pi'_Q$ which has to be nonzero by [Bre16, Lemme 2.2] (whose proof remains unchanged if W is any finite-dimensional locally analytic representation of P). As M^{N_Q} is \mathfrak{l}_Q -simple we can dualize and take $\text{Hom}_{\mathfrak{l}_Q}((M^{N_Q})', -)$ to obtain an injection $\pi_Q \hookrightarrow I$ of smooth L_Q -representations. In fact, this gives an injection (hence isomorphism) $\text{Hom}_G(\mathcal{F}_Q^G(M, \pi_Q), \mathcal{F}_P^G(M, \pi)) \hookrightarrow \text{Hom}_{L_Q}(\pi_Q, I)$, which justifies the first equality.

Suppose that σ is an irreducible subrepresentation of $(\text{Ind}_P^G(M^N)' \otimes \pi)^{\text{an}} = \mathcal{F}_P^G(\underline{M}(M^N), \pi)$. Then σ is a constituent of $\mathcal{F}_P^G(\widetilde{M}, \pi)$ for some simple constituent \widetilde{M} of $\underline{M}(M^N)$ in \mathcal{O}^P . By the beginning of the proof there exists a smooth character η of L such that $\widetilde{M}\eta$ is equimaximal, and we denote its maximal parabolic by $\widetilde{Q} = L_{\widetilde{Q}}N_{\widetilde{Q}}$. We deduce that $\sigma \cong \mathcal{F}_{\widetilde{Q}}^G(\widetilde{M}\eta, \pi_{\widetilde{Q}})$ for some irreducible smooth representation $\pi_{\widetilde{Q}}$ of $L_{\widetilde{Q}}$. Then as in [Bre16] we obtain from Proposition 2.30 a nonzero L -equivariant map

$$\begin{aligned} M^N \otimes (\pi')^N &\xrightarrow{(2.3)} H^0(N, \mathcal{F}_{\widetilde{Q}}^G(\widetilde{M}\eta, \pi_{\widetilde{Q}})') = H^0(N \cap L_{\widetilde{Q}}, (\widetilde{M}\eta)^{N_{\widetilde{Q}}} \otimes \pi'_{\widetilde{Q}}) \\ &= (\widetilde{M}\eta)^N \otimes (\pi'_{\widetilde{Q}})^{N \cap L_{\widetilde{Q}}} = \widetilde{M}^N \otimes \eta(\pi'_{\widetilde{Q}})^{N \cap L_{\widetilde{Q}}}. \end{aligned}$$

In particular, $M^N \cong \widetilde{M}^N$ as \mathfrak{l}_C -modules, but this implies $\widetilde{M} \cong M$ in \mathcal{O}^P (only M^N contains the highest weight for the action of \mathfrak{t}'). Therefore, σ has to lie in the image of the map $\mathcal{F}_P^G(M, \pi) \hookrightarrow \mathcal{F}_P^G(\underline{M}(M^N), \pi)$. \square

The following corollary generalizes [Bre16, Corollary 2.7].

Corollary 2.33. *Suppose $M_1, M_2 \in \mathcal{O}^B$ are \mathfrak{g}_C -simple and equimaximal. Let $P_i = L_i N_i$ denote the maximal parabolic for M_i and suppose that π_i is a smooth representation of L_i of finite length. Then we have $\mathcal{F}_{P_1}^G(M_1, \pi_1) \cong \mathcal{F}_{P_2}^G(M_2, \pi_2)$ if and only if $P_1 = P_2$ and there is a smooth character η of $L_1 = L_2$ such that $M_1 \otimes \eta \cong M_2$ and $\pi_1 \otimes \eta \cong \pi_2$.*

Proof. By Proposition 2.30 we have

$$H^0(U, \mathcal{F}_{P_i}^G(M_i, \pi_i)) = H^0(U \cap L_i, M_i^{N_i} \otimes \pi_i') \cong M_i^U \otimes (\pi_i')^{U \cap L_i}.$$

We deduce that $M_1^U|_{\mathfrak{s}_C} \cong M_2^U|_{\mathfrak{s}_C}$, hence M_1, M_2 have the same highest weight in $(\mathfrak{t}')^*$. In particular, $P_1 = P_2$, and we henceforth denote it by $P = LN$. The isomorphism of L -representations $M_1^N \otimes \pi_1' \cong M_2^N \otimes \pi_2'$ shows that $M_1^N|_{\mathfrak{l}_C} \cong M_2^N|_{\mathfrak{l}_C}$, so as in the proof of Lemma 2.22(ii) we deduce that $M_1 \otimes \eta \cong M_2$ for some smooth character η of L . Applying $\text{Hom}_{\mathfrak{l}_C}((M_1^N)', -)$ to the dual isomorphism $(M_1^N)' \otimes \pi_1 \cong (M_1^N \otimes \eta)' \otimes \pi_2$ we get $\pi_1 \cong \pi_2 \otimes \eta^{-1}$. The converse is clear. \square

We also have a weak version when we drop the equimaximality condition.

Corollary 2.34. *Suppose $M_i \in \mathcal{O}^{P_i}$ is \mathfrak{g}_C -simple and π_i is a smooth representation of L_i of finite length ($i = 1, 2$). Let $Q_i \supset P_i$ be maximal such that $M_i \in \mathcal{O}^{Q_i}$. Suppose that $\mathcal{F}_{P_1}^G(M_1, \pi_1), \mathcal{F}_{P_2}^G(M_2, \pi_2)$ share at least one irreducible constituent. Then $Q_1 = Q_2$ and there exists a smooth character $\eta: L_1 \cap L_2 \rightarrow C^\times$ such that $M_1 \otimes \eta \cong M_2$ in $\mathcal{O}^{P_1 \cap P_2}$. In particular, $M_1 \cong M_2$ in \mathcal{O}^b .*

Proof. We take a z -extension $1 \rightarrow \underline{T} \rightarrow \underline{\tilde{G}} \rightarrow \underline{G} \rightarrow 1$, as in the proof of Corollary 2.31. We keep the notation of that proof. By Proposition 2.26 for each i there exists a smooth character $\tilde{\eta}_i$ of \tilde{L}_i such that $\tilde{M}_i \tilde{\eta}_i$ is equimaximal, i.e. lies in $\mathcal{O}^{\tilde{Q}_i}$. The inflation of $\mathcal{F}_{P_i}^G(M_i, \pi_i)$ to \tilde{G} becomes

$$\mathcal{F}_{\tilde{P}_i}^{\tilde{G}}(\tilde{M}_i, \tilde{\pi}_i) \cong \mathcal{F}_{\tilde{Q}_i}^{\tilde{G}}(\tilde{M}_i \tilde{\eta}_i, (\text{Ind}_{\tilde{P}_i \cap \tilde{L}_{Q_i}}^{\tilde{L}_{Q_i}} \tilde{\pi}_i \tilde{\eta}_i)^{\text{sm}}).$$

By assumption and Theorem 2.16 we deduce from Corollary 2.33 that $\tilde{Q}_1 = \tilde{Q}_2$ and $\tilde{M}_1 \tilde{\eta}_1 \tilde{\eta}_3 \cong \tilde{M}_2 \tilde{\eta}_2$ in $\mathcal{O}^{\tilde{Q}_1} = \mathcal{O}^{\tilde{Q}_2}$ for some smooth character $\tilde{\eta}_3$ of $\tilde{L}_{Q_1} = \tilde{L}_{Q_2}$. In particular, $\tilde{\eta} := \tilde{\eta}_1 \tilde{\eta}_3 \tilde{\eta}_2^{-1}$ is trivial on T , so descends to a smooth character η of $L_1 \cap L_2$. It follows that $M_1 \otimes \eta \cong M_2$ in $\mathcal{O}^{P_1 \cap P_2}$. \square

Recall that if W is a finite-dimensional locally analytic P -representation on which \mathfrak{t}' acts diagonalizably and π an admissible smooth representation of L , then we have a pairing

$$\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, \pi)}: (U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W) \times (\text{Ind}_P^G W' \otimes \pi)^{\text{an}} \rightarrow C^{\text{an}}(G, \pi),$$

which is used to define $\mathcal{F}_P^G(M, \pi)$ for $M \in \mathcal{O}^P$ [OS14b, §3.8] (recalled in Appendix A). Recall that $\mathcal{F}_P^G(U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W, \pi) = (\text{Ind}_P^G W' \otimes \pi)^{\text{an}}$.

The following lemma generalizes [Bre16, Lemma 3.1].

Lemma 2.35. *Suppose that $P = LN$, $M \in \mathcal{O}^P$, and π an admissible smooth L -representation. Suppose we are given morphisms $W_1 \rightarrow M^N \leftarrow W_2$ in \mathcal{O}^L so that we have corresponding diagrams in \mathcal{O}^P , respectively locally analytic representations of G :*

$$\begin{array}{ccc} U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W_1 & & (\text{Ind}_P^G W'_1 \otimes \pi)^{\text{an}} \\ & \searrow \phi_1 & \swarrow \\ & M & \mathcal{F}_P^G(M, \pi) \\ & \nearrow \phi_2 & \swarrow \\ U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W_2 & & (\text{Ind}_P^G W'_2 \otimes \pi)^{\text{an}} \end{array}$$

Let $f \in \mathcal{F}_P^G(M, \pi)$ with images $h_i \in (\text{Ind}_P^G W'_i \otimes \pi)^{\text{an}}$ and $x_i \in U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W_i$ such that $\phi_1(x_1) = \phi_2(x_2)$ in M . Then $\langle x_1, h_1 \rangle_{C^{\text{an}}(G, \pi)} = \langle x_2, h_2 \rangle_{C^{\text{an}}(G, \pi)}$ in $C^{\text{an}}(G, \pi)$.

Proof. By considering $W_1 \oplus W_2 \rightarrow M^N$ we may reduce to the case where there exists a map $\theta: W_1 \rightarrow W_2$ inducing commutative diagrams

$$\begin{array}{ccccc} W_1 & & U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W_1 & & (\text{Ind}_P^G W'_1 \otimes \pi)^{\text{an}} \\ & \searrow & \downarrow 1 \otimes \theta & \searrow & \swarrow \\ \theta \downarrow & & & M & \mathcal{F}_P^G(M, \pi) \\ & \nearrow & U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W_2 & & \swarrow \\ W_2 & & & & (\text{Ind}_P^G W'_2 \otimes \pi)^{\text{an}} \end{array}$$

Next note that we have

$$\mathcal{F}_P^G(M, \pi) \twoheadrightarrow \mathcal{F}_P^G(\text{im}(\phi_i), \pi) = (\text{Ind}_P^G W'_i \otimes \pi)^{\ker(\phi_i)}$$

by the definition of \mathcal{F}_P^G , i.e. h_i is killed by $\ker(\phi_i)$ with respect to the pairing $\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, \pi)}$. By assumption, $x_2 - (1 \otimes \theta)(x_1) \in \ker(\phi_2)$, so we may replace x_2 by

$(1 \otimes \theta)(x_1)$. But then the lemma comes down to the claim that $\langle x_1, \theta^*(h_2) \rangle_{C^{\text{an}}(G, \pi)} = \langle (1 \otimes \theta)(x_1), h_2 \rangle_{C^{\text{an}}(G, \pi)}$, which is obvious from the definitions. \square

The following proposition generalizes [Bre16, Proposition 3.2].

Proposition 2.36. *Suppose that $P = LN$, $M \in \mathcal{O}^P$ \mathfrak{g}_C -simple, and π an (admissible) smooth L -representation of finite length. Let $f \in (\text{Ind}_P^G(M^N)' \otimes \pi)^{\text{an}}$ such that the restriction $f|_{\overline{N}}$ is locally constant. Then $f \in (\text{Ind}_P^G(M^N)' \otimes \pi)^{\ker(\phi)} = \mathcal{F}_P^G(M, \pi)$, where ϕ is the natural map $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} M^N \rightarrow M$.*

Proof. The proof proceeds exactly as in [Bre16], noting that by Lemma 2.11 we know that $W := M^N = M^{\text{nc}}$ is in \mathcal{O}^L and \mathfrak{l}_C -simple. We may assume $f \neq 0$. Note that $\Delta_f = \langle \cdot, f \rangle_{C^{\text{an}}(G, \pi)}: U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} W \rightarrow C^{\text{an}}(G, \pi)$ is also P -linear, by letting P act on $C^{\text{an}}(G, \pi)$ by $(p'f)(g) := p'f(p'^{-1}g)$. Now $\ker(\Delta_f)$ is (\mathfrak{g}_C, P) -stable, and hence a subobject in \mathcal{O}^P , so also $\text{im}(\Delta_f)$ lies in \mathcal{O}^P . As $f \in \mathcal{F}_P^G(\text{im}(\Delta_f), \pi)$ by construction, it suffices to show that $M' := \ker(\text{im}(\Delta_f) \rightarrow M) \in \mathcal{O}^P$ is zero. If not, then $V := M'^N = M'^{\text{nc}}$ is nonzero in \mathcal{O}^L (it need not be simple, but it does not matter for the argument). Then let $\psi: U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} V \rightarrow \text{im}(\Delta_f)$ be the induced map in \mathcal{O}^P . Following the proof [Bre16, Proposition 3.2], we use Lemma 2.35 instead of [Bre16, Lemma 3.1]. For the argument with weights near the end, we may use weights of \mathfrak{t}' . \square

2.6. Intertwiners. Suppose that $\underline{P} = \underline{L}\underline{N}$ be a parabolic subgroup and σ, τ are continuous representations of L on Banach spaces.

Proposition 2.37. *The natural map $\text{Hom}_L^{\text{cts}}(\sigma, \tau) \rightarrow \text{Hom}_G^{\text{cts}}((\text{Ind}_P^G \sigma)^{\text{cts}}, (\text{Ind}_P^G \tau)^{\text{cts}})$ is an isomorphism.*

When \underline{G} is split, \underline{P} is minimal, and $\dim_C \sigma = \dim_C \tau = 1$, then it follows from the main theorem of [BH21].

Proof. By Frobenius reciprocity

$$\text{Hom}_G^{\text{cts}}((\text{Ind}_P^G \sigma)^{\text{cts}}, (\text{Ind}_P^G \tau)^{\text{cts}}) \cong \text{Hom}_P^{\text{cts}}((\text{Ind}_P^G \sigma)^{\text{cts}}, \tau).$$

Fix a nonzero element $\mu \in \text{Hom}_P^{\text{cts}}((\text{Ind}_P^G \sigma)^{\text{cts}}, \tau)$.

Recall that we fixed a special point x_0 in the apartment of \underline{S} corresponding to K . Let I be an Iwahori subgroup fixing a facet of the apartment of \underline{S} having vertex x_0 . Then $G = \coprod_{w \in W_L \backslash W} PwI$ where W_L is the Weyl group $N_{\underline{L}}(\underline{S})/\underline{Z}$. Then $(\text{Ind}_P^G \sigma)^{\text{cts}} \cong \bigoplus_w (\text{Ind}_P^{PwI} \sigma)^{\text{cts}}$ and $(\text{Ind}_P^{PwI} \sigma)^{\text{cts}} \cong \mathcal{C}^0(I \cap w^{-1}\overline{N}w, \sigma)$. We first claim that the restriction of μ to $(\text{Ind}_P^{PwI} \sigma)^{\text{cts}}$ is zero for all $w \in W \setminus W_L$. If $w \notin W_L$ there exists a reduced root $\alpha > 0$ that appears in $\text{Lie}(\underline{N})$ such that $w\alpha$ appears in $\text{Lie}(\overline{N})$, so $U_\alpha \subset N \cap w^{-1}\overline{N}w$. (The argument is exactly as in the proof of Proposition 2.30.) Note that $I \cap w^{-1}\overline{N}w = \prod_{\beta} (I \cap U_{w^{-1}\beta})$ in any fixed order, with β running through the roots of \overline{N} , and we list $\beta = w\alpha$ first. Then Lemma 2.38 shows that any continuous $I \cap U_\alpha$ -linear map $\mathcal{C}^0(I \cap w^{-1}\overline{N}w, \sigma) \rightarrow \tau$ vanishes. This implies the claim, as $I \cap U_\alpha \subset I \cap N \cap w^{-1}\overline{N}w$.

By the claim, the given map μ factors as

$$(\text{Ind}_P^G \sigma)^{\text{cts}} \twoheadrightarrow (\text{Ind}_P^{PI} \sigma)^{\text{cts}} \cong \mathcal{C}^0(I \cap \overline{N}, \sigma) \rightarrow \tau,$$

where the first map is given by restriction. By the action of L , μ also factors through $(\text{Ind}_P^G \sigma)^{\text{cts}} \rightarrow (\text{Ind}_P^{P\ell} \sigma)^{\text{cts}} \cong \mathcal{C}^0(\ell^{-1}I\ell \cap \overline{N}, \sigma)$ for any $\ell \in L$. In other words, μ factors through $\mathcal{C}^0(\overline{N}_0, \sigma)$ for any compact open subgroup \overline{N}_0 of \overline{N} . Fix now some compact open subgroup \overline{N}_0 of \overline{N} and $v \in \sigma$. Write $1_{\overline{N}_0, v} \in (\text{Ind}_B^{B\overline{N}_0} \sigma)^{\text{cts}}$ for the element taking constant value v on \overline{N}_0 . Then $\mu(1_{\ell\overline{N}_0\ell^{-1}, \ell v}) = \mu(\ell 1_{\overline{N}_0, v}) = \ell\mu(1_{\overline{N}_0, v})$ for $\ell \in L$ by L -linearity. But we also have $\mu(1_{\overline{N}_0, \ell v}) = \mu(1_{\ell\overline{N}_0\ell^{-1}, \ell v})$, as

μ only depends on the restriction to a small neighborhood of $1 \in \overline{N}$. Defining $\phi_\mu : \sigma \rightarrow \tau$ by $\phi_\mu(v) := \mu(1_{\overline{N}_0, v})$, we deduce that f is L -linear. It is clearly continuous. Moreover, $\mu(f) = \phi_\mu(f(1))$ for all $f \in \mathcal{C}^\infty(\overline{N}_0, \sigma)$ and hence by continuity for all $f \in \mathcal{C}^0(\overline{N}_0, \sigma)$. It is clear that $\mu \mapsto \phi_\mu$ is inverse to the given map. \square

Lemma 2.38. *Suppose that H is a compact locally analytic group with closed subgroups H_1, H_2 such that multiplication induces a topological isomorphism $H_1 \times H_2 \xrightarrow{\sim} H$. Suppose there exist a basis of open neighborhoods of 1 consisting of subgroups of the form $H'_1 H'_2$ with H'_i an open subgroup of H_i . Suppose that H_1 is infinite and that V_1, V_2 are Banach spaces. Then any left H_1 -invariant continuous map $\mu : \mathcal{C}^0(H, V_1) \rightarrow V_2$ vanishes.*

Proof. Suppose that μ is nonzero. By density of smooth functions and by assumption, μ has to be nonzero on a function of the form $1_{U, h, v}$ for some open subgroup U , some $h \in H$, and $v \in V_1$. Without loss of generality, $U = H'_1 H'_2$ with H'_i an open subgroup of H_i . Then for any open subgroup H''_1 of H'_1 we have $\mu(1_{H''_1 H'_2, g, v}) = (H'_1 : H''_1)^{-1} \mu(1_{H'_1 H'_2, g, v})$ by left H_1 -invariance. By assumption on H_1 we deduce that the p -adic absolute value of $(H'_1 : H''_1)^{-1}$ is unbounded. (Note that H'_1 is compact locally analytic, hence profinite and contains an open normal pro- p subgroup, which is infinite by assumption.) This contradicts the continuity of μ , as the set $\{1_{H''_1 H'_2, g, v}\}$ is bounded. \square

Corollary 2.39. *If σ is indecomposable then $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is indecomposable.*

Here, we say that a Banach representation π is *indecomposable* if it cannot be written as a direct sum of two closed subrepresentations. (Equivalently, the ring $\text{End}_G^{\text{cts}}(\pi)$ does not contain any non-trivial idempotents.)

We say that an admissible Banach representation is *semisimple* if it is a direct sum of finitely many simple subobjects (i.e. irreducible closed subrepresentations).

Lemma 2.40. *Let π be a finite-length admissible Banach representation. Then the following are equivalent.*

- (i) π is semisimple.
- (ii) π is a finite sum of irreducible subrepresentations.
- (iii) For any closed subrepresentation $\sigma \subset \pi$ there exists a closed subrepresentation τ such that $\pi = \sigma \oplus \tau$.

In particular, any closed subrepresentation of a semisimple representation is semisimple.

Proof. The proof works in any abelian category. Assume that $\pi = \sum_{i=1}^n \pi_i$ such that π_i are simple subobjects and n is minimal. If the sum is not direct there exists $1 \leq j \leq n$ such that $\pi_j \cap \sum_{i \neq j} \pi_i \neq 0$, i.e. $\pi_j \subset \sum_{i \neq j} \pi_i$, contradicting the minimality of n . Therefore (ii) implies (i), and the converse is obvious.

Assume π is semisimple and write $\pi = \bigoplus_{i=1}^n \pi_i$ with π_i simple. Let $\sigma \subset \pi$ be a subobject. Pick a sequence $1 \leq k_1 < k_2 < \dots < k_s \leq n$ of minimal length such that $\sigma + \sum_{j=1}^s \pi_{k_j} = \pi$. If the sum is not direct, then $\pi_{k_{j'}} \cap (\sigma + \sum_{j \neq j'} \pi_{k_j}) \neq 0$ for some $1 \leq j' \leq s$, i.e. $\pi_{k_{j'}} \subset \sigma + \sum_{j \neq j'} \pi_{k_j}$, which contradicts the minimality of s . Hence we can take $\tau = \sum_{j=1}^s \pi_{k_j}$. Therefore (i) implies (iii), and the converse follows by induction on the length of π .

Any subobject of a semisimple representation π is also a quotient of π by (iii) and hence satisfies (ii). \square

If C'/C is a finite extension, then admissibility and semisimplicity of Banach representations is preserved under extension of scalars from C to C' and under

restriction of scalars from C' to C . (Note that if π is an admissible Banach representation over C' and π^0 denotes an $\mathcal{O}_{C'}$ -stable unit ball in π , then $\pi^0/\varpi_{C'}\pi^0$ is admissible smooth if and only if $\pi^0/\varpi_C\pi^0$ is admissible smooth, so π is admissible over C . For semisimplicity we can argue using Lemma 2.40, picking an irreducible subrepresentation of the extension/restriction first. In the case of extension of scalars it helps to reduce to the case where C'/C is finite Galois, cf. also [DS13, Lemma 3.7].)

Lemma 2.41. *Suppose that G is a locally analytic group with open normal subgroup N such that G/N is finite abelian. If V is an absolutely irreducible admissible Banach representation of G and C is sufficiently large, then there exists an absolutely irreducible closed subrepresentation W of $V|_N$ such that, if we let $H := \{g \in G \mid gW = W\}$ (stabilizer of the subspace), we have $W \circ \text{Ad}(g) \cong W$ as H -representations for $g \in G$ implies $g \in H$. In particular, the natural map $\text{Ind}_H^G W \rightarrow V$ is an isomorphism.*

Proof. (See [Roc09, 1.6.3] in the context of smooth representations.) By Clifford theory [BS19, Proposition 2.1.1], $V|_N$ is a direct sum of finitely many irreducible subrepresentations. So as C is sufficiently large, we may assume that any irreducible closed subrepresentation of $V|_N$ is absolutely irreducible. Choose now W among all irreducible closed subrepresentations of $V|_N$ such that $H := \{g \in G \mid gW = W\}$ is maximal. Note that H is contained in $\tilde{H} := \{g \in G \mid W \circ \text{Ad}(g) \cong W \text{ as } H\text{-representations}\}$, and if $H = \tilde{H}$, then we are done. If not, pick $H \subsetneq H_1 \subset \tilde{H}$ such that H_1/H is cyclic. As C is sufficiently large, W extends to an irreducible representation \tilde{W} of H_1 . From $\text{Hom}_H(W, V|_H) \cong \text{Hom}_{H_1}(\text{Ind}_H^{H_1} W, V|_{H_1})$ and $\text{Ind}_H^{H_1} W \cong \bigoplus_{\eta: H_1/H \rightarrow C^\times} \tilde{W} \otimes \eta$ we deduce that some extension $\tilde{W} \otimes \eta$ of W occurs as closed subrepresentation of $V|_{H_1}$. (Note that the image of any morphism of admissible Banach representations is closed.) This contradicts the maximality of H . \square

Proposition 2.42. *Let $\varphi: \underline{G}_1 \rightarrow \underline{G}$ be a morphism such that $\varphi(\underline{G}_1^{\text{der}}) = \underline{G}^{\text{der}}$ and $\ker \varphi \subset \underline{Z}_{\underline{G}_1}$. Suppose that $\underline{P} = \underline{LN}$ is a parabolic subgroup of \underline{G} and let $\underline{P}_1 := \varphi^{-1}(\underline{P})$, $\underline{L}_1 := \varphi^{-1}(\underline{L})$, $\underline{N}_1 := \varphi^{-1}(\underline{N})$. Suppose that σ is an admissible Banach representation of L , and write σ_1 for the composition of σ with $L_1 \rightarrow L$. Then we have the following.*

- (i) *If σ is irreducible or it has a central character, then σ_1 is admissible.*
- (ii) *If $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \sigma_1)^{\text{cts}}$ is irreducible, then $(\text{Ind}_{\underline{P}}^{\underline{G}} \sigma)^{\text{cts}}$ and σ_1 are irreducible.*
- (iii) *If $(\text{Ind}_{\underline{P}}^{\underline{G}} \sigma)^{\text{cts}}$ is irreducible, then $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \tau_1)^{\text{cts}}$ is irreducible for any irreducible closed subrepresentation τ_1 of σ_1 .*
- (iv) *Assume that σ is absolutely irreducible. If $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \tau_1)^{\text{cts}}$ is absolutely irreducible for one (equivalently any) absolutely irreducible closed subrepresentation τ_1 of σ_1 , then $(\text{Ind}_{\underline{P}}^{\underline{G}} \sigma)^{\text{cts}}$ is absolutely irreducible.*

Remark 2.43. In particular, if $\dim_C \sigma = 1$, then $(\text{Ind}_{\underline{P}}^{\underline{G}} \sigma)^{\text{cts}}$ is irreducible if and only if $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \sigma_1)^{\text{cts}}$ is irreducible. A special case of this statement can be found in [BS19, Proposition 2.3.2].

Proof. Set $G'_1 := \varphi(G_1)$, a closed locally analytic subgroup of G . (It is closed by [BT73, 3.19 Proposition].) We observe that $Z_G G'_1$ is open and normal in G with finite abelian quotient group. (This follows from [BT73, 3.20 Corollary] applied to the surjective homomorphism $\underline{G}_1 \times \underline{Z}_{\underline{G}}^\circ \rightarrow \underline{G}$.)

We justify that σ_1 is admissible. We put $L'_1 := \varphi(L_1)$. Note that $\varphi(\underline{L}_1^{\text{der}}) = \underline{L}^{\text{der}}$, so we have a surjective homomorphism $\underline{L}_1 \times \underline{Z}_{\underline{L}}^\circ \rightarrow \underline{L}$, which is therefore surjective

on Lie algebras and hence induces an open homomorphism on F -points. Pick any compact open subgroups H of L_1 and H_Z of Z_L° . Then $\varphi(H)H_Z$ is a compact open subgroup of L , so there exists a $\varphi(H)H_Z$ -stable unit ball σ^0 in σ and its reduction is admissible smooth as $\varphi(H)H_Z$ -representation (as σ is admissible). If σ has a central character, then, taking H_Z small enough so that it acts trivially on σ , the reduction of σ^0 is admissible as $\varphi(H)$ -representation, i.e. σ_1 admissible [Eme17, Proposition 6.5.7]. If σ is irreducible, then the endomorphism algebra $\text{End}_L^{\text{cts}}(\sigma)$ is a finite-dimensional division algebra [DS13]. Let C' be the subalgebra of $\text{End}_L^{\text{cts}}(\sigma)$ generated by the image of Z_L , which is a finite extension of C . Then σ is admissible as Banach representation of L over C' and has a central character, so σ_1 is admissible over C' , so σ_1 is admissible over C .

The group G_1 acts on $(\text{Ind}_{P_1}^{G_1} \sigma_1)^{\text{cts}}$ through $G_1 \twoheadrightarrow G'_1$ and as G'_1 -representations we have $(\text{Ind}_{P_1}^{G_1} \sigma_1)^{\text{cts}} \cong (\text{Ind}_P^G \sigma)^{\text{cts}}|_{G'_1}$. (To see this, we note that φ induces an isomorphism $P_1 \backslash G_1 \xrightarrow{\sim} P \backslash G$ because it induces an isomorphism $P_1 \backslash \underline{G}_1 \xrightarrow{\sim} P \backslash \underline{G}$ and we obtain the former isomorphism by passing to F -points.) Hence if $(\text{Ind}_{P_1}^{G_1} \sigma_1)^{\text{cts}}$ is irreducible then $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is also irreducible. Moreover σ_1 is clearly irreducible.

Conversely, assume that $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible. Then in particular σ is irreducible and again $\text{End}_L^{\text{cts}}(\sigma)$ is a finite-dimensional division algebra. Let C' be the subfield of $\text{End}_L^{\text{cts}}(\sigma)$ generated by the image of Z_G . Considering σ as irreducible admissible representation of L over C' , by Clifford theory [BS19, Proposition 2.1.1] $(\text{Ind}_P^G \sigma)^{\text{cts}}|_{Z_G G'_1}$ is semisimple (as defined above) over C' . As σ has a central character over C' , $(\text{Ind}_P^G \sigma)^{\text{cts}}|_{G'_1}$ is semisimple over C' and hence semisimple over C . In particular, by Lemma 2.40 the direct summand $(\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}}$ of $(\text{Ind}_P^G \sigma)^{\text{cts}}|_{G'_1}$ is semisimple over C . But it is also indecomposable by Corollary 2.39, hence irreducible.

Finally assume that σ is absolutely irreducible. As we have seen before (for $\underline{G} = \underline{L}$), $Z_L L'_1$ is open and normal in L with finite abelian quotient group. By Lemma 2.41, applied with $N = Z_L L'_1$ and extending scalars if necessary, we can take an absolutely irreducible closed subrepresentation τ of $\sigma|_{Z_L L'_1}$ such that $H := \{g \in L \mid g\tau = \tau\}$ (stabilizer of the subspace) is equal to $\{g \in L \mid \tau \circ \text{Ad}(g) \cong \tau \text{ as } H\text{-representations}\}$, so $\sigma \cong \text{Ind}_H^L \tau$. Since σ has a central character, τ is also absolutely irreducible as L'_1 -representation. Let τ_1 be the composition of τ with $L_1 \twoheadrightarrow L'_1 \subset H$, which is again absolutely irreducible. Then the subspace $W := \{f \in (\text{Ind}_P^G \sigma)^{\text{cts}} \mid f(Z_G G'_1) \subset \tau\}$ is G'_1 -stable and isomorphic to $(\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}}$. Note that if $\ell \in L$ and $f \in (\text{Ind}_P^G \sigma)^{\text{cts}}$ such that $f(Z_G G'_1) \subset \tau$, then for $g \in Z_G G'_1$ we have $(\ell f)(g) = f(g\ell) = f(\ell(\ell^{-1}g\ell)) = \ell f(\ell^{-1}g\ell) \in \ell\tau_1$. In particular, W is also H -stable. Moreover, together with $\sigma = \bigoplus_{\ell \in L/H} \ell\tau$ we deduce that $(\text{Ind}_P^G \sigma)^{\text{cts}} = \bigoplus_{\ell \in L/H} \ell W$. As $H \backslash L \xrightarrow{\sim} H G'_1 \backslash G$, it follows that the natural continuous homomorphism $\text{Ind}_{H G'_1}^G W \rightarrow (\text{Ind}_P^G \sigma)^{\text{cts}}$ is an isomorphism.

We assume $(\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}} \cong W$ is absolutely irreducible. To prove $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is absolutely irreducible, by the previous paragraph it is sufficient to prove that for $g \in G$, if $(\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}} \circ \text{Ad}(g) \cong (\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}}$ as $H G'_1$ -representations, then $g \in H G'_1$. As $G = L G'_1$, we may assume $g \in L$. The map $f \mapsto (x \mapsto f(gxg^{-1}))$ gives an $H G'_1$ -linear isomorphism $(\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}} \circ \text{Ad}(g) \cong (\text{Ind}_{P_1}^{G_1} \tau_1 \circ \text{Ad}(g))^{\text{cts}}$. By Proposition 2.37 any G_1 -linear isomorphism $\Psi: (\text{Ind}_{P_1}^{G_1} \tau_1 \circ \text{Ad}(g))^{\text{cts}} \xrightarrow{\sim} (\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}}$ arises from an L_1 -linear isomorphism $\psi: \tau_1 \circ \text{Ad}(g) \xrightarrow{\sim} \tau_1$. From the relations $\Psi(f)(1) = \psi(f(1))$ and $(hf)(1) = h \cdot f(1)$ for $f \in (\text{Ind}_{P_1}^{G_1} \tau_1)^{\text{cts}}$ and $h \in H$ we deduce that ψ is even H -linear. Therefore, $g \in H$. \square

Remark 2.44. In this way the problem of understanding when a parabolic induction $(\text{Ind}_P^G \sigma)^{\text{cts}}$ (with σ admissible) is absolutely irreducible can be reduced to the case where \underline{G} is a simply-connected group. If σ is finite-dimensional we may reduce further, see Remark 2.60 below.

2.7. Translation functors. In this subsection we recall translation functors and extend some properties to locally analytic representations. This topic is also studied in [JLS21] for split reductive groups, from the point of view of distribution algebras. (In particular, in [JLS21, Theorem 4.2.12] they compute the effect of translation functors on locally analytic principal series.) We do not refer to this paper and give proofs for the sake of completeness.

We first recall some facts about translation functors on \mathfrak{g}_C -modules from [BG80]. Let $Z(\mathfrak{g}_C)$ be the center of $U(\mathfrak{g}_C)$. We say that a \mathfrak{g}_C -module M is $Z(\mathfrak{g}_C)$ -finite if the ideal $\text{Ann}_{Z(\mathfrak{g}_C)} M \subset Z(\mathfrak{g}_C)$ has finite codimension. Let \mathcal{M}_{Zf} be the category of $Z(\mathfrak{g}_C)$ -finite \mathfrak{g}_C -modules. By the Harish-Chandra isomorphism we have an embedding $Z(\mathfrak{g}_C) \hookrightarrow \text{Sym}(\mathfrak{t}')$ whose image is the set of vectors fixed by the absolute Weyl group. For $\lambda \in (\mathfrak{t}')^*$, let $\chi_\lambda: Z(\mathfrak{g}_C) \rightarrow C$ be the composition of the Harish-Chandra isomorphism and the evaluation at λ . We say that a \mathfrak{g}_C -module M admits generalized infinitesimal character λ if $(\ker \chi_\lambda)^n M = 0$ for some $n \in \mathbb{Z}_{>0}$ and let \mathcal{M}_λ be the full subcategory of \mathcal{M}_{Zf} consisting of all M which admit a generalized infinitesimal character λ . (We note that our definitions are slightly more restrictive than the ones used in [BG80].)

In this subsection, we say that $\lambda \in (\mathfrak{t}')^*$ is *integral* if its value at any absolute coroot is an integer. Note that this is weaker than $\lambda \in X^*(\underline{T}')$. Let $\lambda, \mu \in (\mathfrak{t}')^*$ and assume that $\mu - \lambda$ is integral. We define the translation functor $T_\lambda^\mu: \mathcal{M}_\lambda \rightarrow \mathcal{M}_\mu$ as follows. Let V be the absolutely simple finite-dimensional \mathfrak{g}_C -module having $\mu - \lambda$ as an extremal weight. By a theorem of Kostant, we have $V \otimes \mathcal{M}_{Zf} \subset \mathcal{M}_{Zf}$ [BG80, 2.6 Corollary] (and its proof). Then we define $T_\lambda^\mu(X) := \text{pr}_\mu(V \otimes X)$, where $\text{pr}_\mu: \mathcal{M}_{Zf} \rightarrow \mathcal{M}_\mu$ is the projection. The properties of translation functors are summarized as follows.

Proposition 2.45. *Let $\lambda, \mu \in (\mathfrak{t}')^*$ such that $\mu - \lambda$ is integral.*

- (i) *The functor T_λ^μ is exact.*
- (ii) *The pair $(T_\lambda^\mu, T_\mu^\lambda)$ is an adjoint pair.*
- (iii) *Assume that for any absolute root α we have $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$ if and only if $\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. Then T_λ^μ gives an equivalence of categories.*

Proof. (i) and (ii) are easy and (iii) is [BG80, 4.1 Theorem]. \square

We upgrade the above constructions to locally analytic representations. Let $\mathcal{M}_\lambda^{\text{an}} = \mathcal{M}_\lambda^{\text{an}}(G)$ be the category of locally analytic representations π of G such that $\pi|_{\mathfrak{g}_C} \in \mathcal{M}_\lambda$. Assume that there exists a finite-dimensional locally analytic representation V of G such that $V|_{\mathfrak{g}_C}$ is absolutely simple with extremal weight $\mu - \lambda$ and we fix such V . (It exists by Lemma 2.22(ii) if $\underline{G}^{\text{der}}$ is simply connected.) Then for $X \in \mathcal{M}_\lambda^{\text{an}}$, $V \otimes X$ is also a locally analytic representation of G . Since $\text{pr}_\mu(V \otimes X)$ is the kernel of certain $z \in Z(\mathfrak{g}_C)$, it is a closed subspace. Moreover it is G -stable since the action of G commutes with that of $Z(\mathfrak{g}_C)$, as $Z(\mathfrak{g}_C) = U(\mathfrak{g}_C)^{\text{ad}(\mathfrak{g}_C)} = U(\mathfrak{g}_C)^{\text{Ad}(G_C)}$. (Note that the action on $U(\mathfrak{g}_C)$ is locally finite-dimensional algebraic.) Therefore we can define the functor $T_\lambda^\mu(G, V): \mathcal{M}_\lambda^{\text{an}} \rightarrow \mathcal{M}_\mu^{\text{an}}$ by $T_\lambda^\mu(G, V)(X) := \text{pr}_\mu(V \otimes X)$.

Lemma 2.46. *Suppose that π is a locally analytic representation such that $\pi|_{\mathfrak{g}_C}$ is $Z(\mathfrak{g}_C)$ -finite. For any $\mu \in (\mathfrak{t}')^*$ there exists a functorial decomposition $\pi \cong \text{pr}_\mu(\pi) \oplus \text{pr}'_\mu(\pi)$ as locally analytic representations of G .*

Proof. By assumption there exists an ideal I of $Z(\mathfrak{g}_C)$ of finite codimension such that $I\pi = 0$. Then $Z(\mathfrak{g}_C)/I$ is an artinian ring and we have a (finite) decomposition $\pi = \bigoplus_{\mathfrak{m}} \pi_{\mathfrak{m}}$, where \mathfrak{m} runs through the maximal ideals of $Z(\mathfrak{g}_C)$. We define $\text{pr}'_{\mu}(\pi) := \bigoplus_{\mathfrak{m} \neq \ker \chi_{\mu}} \pi_{\mathfrak{m}}$. As above, each $\pi_{\mathfrak{m}}$ is a closed G -equivariant subspace of π and the projection $\pi \rightarrow \pi_{\mathfrak{m}}$ is continuous. We obtain continuous bijections $\text{pr}_{\mu}(\pi) \oplus \text{pr}'_{\mu}(\pi) \rightarrow \pi \rightarrow \text{pr}_{\mu}(\pi) \times \text{pr}'_{\mu}(\pi)$, which are topological isomorphisms since the composition is one. \square

Proposition 2.47. *Let $\lambda, \mu \in (\mathfrak{t}')^*$ such that $\mu - \lambda$ is integral.*

- (i) *The functor $T_{\lambda}^{\mu}(G, V)$ preserves strict exact sequences.*
- (ii) *The pair $(T_{\lambda}^{\mu}(G, V), T_{\mu}^{\lambda}(G, V'))$ is an adjoint pair.*
- (iii) *Assume that for any absolute root α we have $\langle \lambda, \alpha^{\vee} \rangle \in \mathbb{Z}_{>0}$ if and only if $\langle \mu, \alpha^{\vee} \rangle \in \mathbb{Z}_{>0}$. Then $T_{\lambda}^{\mu}(G, V): \mathcal{M}_{\lambda}^{\text{an}} \rightarrow \mathcal{M}_{\mu}^{\text{an}}$ gives an equivalence of categories.*

Proof. For (i), by Proposition 2.45(i) it suffices to show that T_{λ}^{μ} sends strict morphisms to strict morphisms. In fact, the same is true for $X \mapsto V \otimes X$ and $X \mapsto \text{pr}_{\mu}(X)$. This follows from the basic fact that if $f_i: \pi_i \rightarrow \pi'_i$ ($i = 1, \dots, n$) in the category of locally convex spaces, then f_i is strict for all i if and only if $\bigoplus f_i: \bigoplus_{i=1}^n \pi_i \rightarrow \bigoplus_{i=1}^n \pi'_i$ is strict. For (ii), recall that units and counits of $(T_{\lambda}^{\mu}(G, V), T_{\mu}^{\lambda}(G, V'))$ on $\mathcal{M}_{\lambda}, \mathcal{M}_{\mu}$ are induced by $C \rightarrow V' \otimes V$ and $V' \otimes V \rightarrow C$. Both are G -equivariant linear morphisms, so the unit and the counit are morphisms of locally analytic representations. Part (iii) follows from (ii) and Proposition 2.45 (iii). \square

For the following result, recall that in §2.4 we fixed a system of positive roots for $(\mathfrak{g}_C, \mathfrak{t}')$. Let $\rho \in (\mathfrak{t}')^*$ (resp. $\rho_L \in (\mathfrak{t}')^*$) be the half sum of positive roots in \mathfrak{g}_C (resp. in \mathfrak{l}_C).

Proposition 2.48. *Let $\underline{P} = \underline{LN}$ be a parabolic subgroup of \underline{G} . Suppose $V \in \mathcal{O}^G$ is \mathfrak{g}_C -simple with lowest weight λ . Let V_N denote the N -coinvariants in V , which is a locally analytic representation of L (in \mathcal{O}^L).*

- (i) *If $\sigma \in \mathcal{M}_{\mu}^{\text{an}}(L)$ for some $\mu \in (\mathfrak{t}')^*$, then $(\text{Ind}_{\underline{P}}^G \sigma)^{\text{an}} \in \mathcal{M}_{\mu - \rho + \rho_L}^{\text{an}}(G)$.*
- (ii) *For any smooth representation τ of L , we have*

$$T_{-\rho}^{\lambda - \rho}(G, V)((\text{Ind}_{\underline{P}}^G \tau)^{\text{an}}) \cong (\text{Ind}_{\underline{P}}^G V_N \otimes \tau)^{\text{an}}.$$

Note that λ is automatically integral. Moreover, the proof shows that the isomorphism in (ii) is obtained by applying $\text{pr}_{\lambda - \rho}$ to the natural surjection $(\text{Ind}_{\underline{P}}^G V \otimes \tau)^{\text{an}} \rightarrow (\text{Ind}_{\underline{P}}^G V_N \otimes \tau)^{\text{an}}$.

Proof. Let $\gamma: Z(\mathfrak{g}_C) \hookrightarrow \text{Sym } \mathfrak{t}'$ and $\gamma_L: Z(\mathfrak{l}_C) \hookrightarrow \text{Sym } \mathfrak{t}'$ be the Harish-Chandra isomorphisms. We also define $\gamma': Z(\mathfrak{g}_C) \rightarrow Z(\mathfrak{l}_C)$ as follows. By considering the adjoint action of the center of \mathfrak{l}_C , we have $Z(\mathfrak{g}_C) \subset U(\mathfrak{l}_C) \oplus \mathfrak{n}_C U(\mathfrak{g}_C) \bar{\mathfrak{n}}_C$ and let γ' be the first projection along this decomposition. It is easy to see that γ' is an \mathfrak{l}_C -bimodule homomorphism, hence $\gamma'(Z(\mathfrak{g}_C)) \subset Z(\mathfrak{l}_C)$. For each $\lambda \in (\mathfrak{t}')^*$, define $t_{\lambda}: \text{Sym } \mathfrak{t}' \rightarrow \text{Sym } \mathfrak{t}'$ by $t_{\lambda}(H) = H + \lambda(H)$ for $H \in \mathfrak{t}'$. Then we have $\gamma = t_{\rho - \rho_L} \circ \gamma_L \circ \gamma'$ by the definitions. (Note that we work here with opposite Borels because the usual Harish-Chandra map is obtained by projecting from $Z(\mathfrak{g}_C) \subset U(\mathfrak{t}') \oplus \bar{\mathfrak{u}}' U(\mathfrak{g}_C) \mathfrak{u}'$, where we recall that \mathfrak{u}' is the unipotent radical of the Borel subalgebra \mathfrak{b}' , and we let $\bar{\mathfrak{u}}'$ denote its opposite.)

We prove (i). For each $u \in U(\mathfrak{g}_C)$, let R_u (resp. L_u) be the right translation (resp. left translation) of u on locally analytic functions on G . Let $f \in (\text{Ind}_{\underline{P}}^G \sigma)^{\text{an}}$ and $z \in Z(\mathfrak{g}_C)$. Then for $g \in G$, we have $(zf)(g) = (R_z f)(g) = (L_{\text{Ad}(g)z} f)(g)$

and this is equal to $(L_z f)(g)$ since $z \in Z(\mathfrak{g}_C) = U(\mathfrak{g}_C)^{\text{Ad}(G_C)}$. As f is left N -invariant, $L_u f = 0$ for $u \in \mathfrak{n}_C U(\mathfrak{g}_C)$. Hence $z f = L_z f = L_{\gamma'(z)} f$. We also have $(L_u f)(g) = \sigma(u) f(g)$ for $u \in U(\mathfrak{l}_C)$. Therefore $(L_{\gamma'(z)} f)(g) = \gamma'(z) f(g)$. As $\gamma = t_{\rho - \rho_L} \circ \gamma_L \circ \gamma'$, if σ is killed by $(\ker \chi_\mu)^n$ for $n \in \mathbb{Z}_{>0}$, then $(\text{Ind}_P^G \sigma)^{\text{an}}$ is killed by $(t_{\rho - \rho_L}(\ker \chi_\mu))^n = (\ker \chi_{\mu - \rho + \rho_L})^n$. Part (i) follows.

By Lemma 2.3, we have $V \otimes (\text{Ind}_P^G \tau)^{\text{an}} \cong (\text{Ind}_P^G V \otimes \tau)^{\text{an}}$. The P -representation $V|_P$ has a filtration such that the successive quotients are \mathfrak{l}_C -simple and N acts trivially (apply Lemma 2.28 with $W = V^N$). Let V_0 be such a subquotient and let $\nu \in (\mathfrak{l}')^*$ be the lowest weight of V_0 . As V_N has the same lowest weight as V , it is sufficient to prove that if $\text{pr}_{\lambda - \rho}((\text{Ind}_P^G V_0 \otimes \tau)^{\text{an}}) \neq 0$ then $\nu = \lambda$.

The representation V_0 has infinitesimal character $\nu - \rho_L$. By (i), $(\text{Ind}_P^G V_0 \otimes \tau)^{\text{an}}$ has infinitesimal character $\nu - \rho$. Therefore if it has infinitesimal character $\lambda - \rho$ then $\lambda - \rho = w(\nu - \rho)$ for an element w in the absolute Weyl group. Then we have $\lambda - w(\nu) = \rho - w(\rho)$. The weight $w(\nu)$ is a weight in V and since λ is lowest weight of V , $\lambda - w(\nu)$ is a non-positive linear combination of positive roots. On the other hand $\rho - w(\rho)$ is a non-negative linear combination of positive roots. Hence both sides are zero. From $\rho = w(\rho)$ we have $w = 1$ and we get $\lambda = w(\nu) = \nu$, as desired. \square

2.8. A criterion. *In this section, we assume $F = \mathbb{Q}_p$. Let $\underline{P} = \underline{LN}$ be a parabolic subgroup of \underline{G} and σ a finite-dimensional absolutely irreducible continuous representation of L .*

Lemma 2.49. *We have $\sigma \in \mathcal{O}^L$, after perhaps replacing C by a finite extension.*

Proof. By [Ser92, §V.9] (and as $F = \mathbb{Q}_p$) σ is in fact a locally analytic representation. If $W \subset \sigma|_{\mathfrak{l}_C}$ is a simple submodule, then by irreducibility, $\sigma = \sum_{x \in L} xW$ and each xW is \mathfrak{l}_C -stable, so $\sigma|_{\mathfrak{l}_C}$ is semisimple. After a finite scalar extension, $\sigma|_{\mathfrak{l}_C}$ is a direct sum of absolutely simple \mathfrak{l}_C -modules, i.e. $\sigma \in \mathcal{O}^L$. \square

Therefore we may assume that σ is an absolutely simple object of \mathcal{O}^L .

We now confirm an expectation of [Sch06, §2] for Banach representations.

Proposition 2.50. *Assume Assumption 2.14. Suppose that σ is a finite-dimensional continuous representation of L . Then the Banach representation $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is admissible and topologically of finite length.*

Proof. The admissibility follows as in [Sch06, Proposition 2.4] (where $\dim_C \sigma = 1$). For the finite length statement, by [ST03, Theorem 7.1] (as $F = \mathbb{Q}_p$) it suffices to show that the admissible locally analytic representation $(\text{Ind}_P^G \sigma)^{\text{an}} \cong \mathcal{F}_P^G(\underline{M}(\sigma'), 1)$ is topologically of finite length, but this is a consequence of Corollary 2.18. \square

Lemma 2.51. *Suppose that $\underline{G}^{\text{der}}$ is simply connected. Then, after perhaps replacing C by a finite extension, we can write $\sigma \cong \sigma_0 \otimes \tau$, where $\sigma_0 \in \mathcal{O}^L$ is \mathfrak{l}_C -simple and τ is an (absolutely irreducible) smooth L -representation such that moreover $\underline{L}(\sigma'_0) \in \mathcal{O}^P$ is equimaximal.*

Proof. By Proposition 2.24(iii) we can write $\sigma \cong \sigma_0 \otimes \tau$, where $\sigma_0 \in \mathcal{O}^L$ is \mathfrak{l}_C -simple and τ is an absolutely irreducible smooth L -representation. By Proposition 2.26 there exists a smooth character η of L such that $\underline{L}(\sigma'_0) \otimes \eta \in \mathcal{O}^P$ is equimaximal. (Both steps may require a finite scalar extension.) By replacing (σ_0, τ) by $(\sigma_0 \eta, \tau \eta^{-1})$ we may assume that $\eta = 1$, i.e. $\underline{L}(\sigma'_0)$ is equimaximal. \square

For the remainder of this subsection we will assume that $\sigma \cong \sigma_0 \otimes \tau$, where $\sigma_0 \in \mathcal{O}^L$ is \mathfrak{l}_C -simple and τ is an absolutely irreducible smooth L -representation such that moreover $\underline{L}(\sigma'_0) \in \mathcal{O}^P$ is equimaximal with maximal parabolic $Q = L_Q N_Q$

(containing P). This decomposition of σ always exists if $\underline{G}^{\text{der}}$ is simply connected, by Lemma 2.51, but not in general. By Lemma 2.22(ii) the decomposition is unique up to a smooth character of L_Q .

Note that

$$\mathcal{F}_{P \cap L_Q}^{L_Q}(\underline{L}_{L_Q}(\sigma'_0), \tau) \subset \mathcal{F}_{P \cap L_Q}^{L_Q}(\underline{M}_{L_Q}(\sigma'_0), \tau) = (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{an}} \subset (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}.$$

We say that σ *satisfies condition (*)* if

any irreducible subrepresentation of $\mathcal{F}_{P \cap L_Q}^{L_Q}(\underline{L}_{L_Q}(\sigma'_0), \tau)$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}$.

Note that this condition does not depend on the choice of factorization $\sigma \cong \sigma_0 \otimes \tau$. Note also that $\underline{L}_{L_Q}(\sigma'_0)$ lies in \mathcal{O}^{L_Q} by Corollary 2.12 and that tensoring with $\underline{L}_{L_Q}(\sigma'_0)'$ gives a correspondence between (closed) subrepresentations of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ and (closed) subrepresentations of

$$\mathcal{F}_{P \cap L_Q}^{L_Q}(\underline{L}_{L_Q}(\sigma'_0), \tau) = \mathcal{F}_{L_Q}^{L_Q}(\underline{L}_{L_Q}(\sigma'_0), (\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}) = \underline{L}_{L_Q}(\sigma'_0)' \otimes (\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}.$$

(The point is that $\underline{L}_{L_Q}(\sigma'_0)'$ is absolutely simple as $\mathfrak{l}_{Q,C}$ -module and that all representations here carry the finest locally convex topology.)

Lemma 2.52. *Assume Assumption 2.14. If any irreducible subrepresentation of $\mathcal{F}_{P \cap L_Q}^{L_Q}(1, \tau) = (\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$, then (*) holds.*

Proof. We may simplify notation by relabeling Q as G , i.e. assume that $\underline{L}(\sigma'_0) \in \mathcal{O}^G$.

Let $V := \underline{L}(\sigma'_0)'$, which is by assumption a finite-dimensional locally analytic representation of G . Note that $\sigma'_0 \hookrightarrow V'$ as P -representations, i.e. $V \rightarrow \sigma_0$, giving by Lemma 2.3 a commutative diagram (where all maps are continuous):

$$\begin{array}{ccccccc} V \otimes (\text{Ind}_P^G \tau)^{\text{sm}} & \hookrightarrow & V \otimes (\text{Ind}_P^G \tau)^{\text{an}} & \xrightarrow{\sim} & (\text{Ind}_P^G V \otimes \tau)^{\text{an}} & \twoheadrightarrow & (\text{Ind}_P^G \sigma)^{\text{an}} \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & V \otimes (\text{Ind}_P^G \tau)^{\text{cts}} & \xrightarrow{\sim} & (\text{Ind}_P^G V \otimes \tau)^{\text{cts}} & \twoheadrightarrow & (\text{Ind}_P^G \sigma)^{\text{cts}} \end{array}$$

We first show that the composition of the top horizontal arrows is injective with image $\mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau)$. By the discussion before this lemma we know that $V \otimes (\text{Ind}_P^G \tau)^{\text{sm}} \cong \mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau)$ and that its irreducible constituents are of the form $\mathcal{F}_G^G(\underline{L}(\sigma'_0), \pi')$ for some irreducible smooth representations π' of G . It thus suffices to show that every irreducible constituent of $\ker((\text{Ind}_P^G V \otimes \tau)^{\text{an}} \twoheadrightarrow (\text{Ind}_P^G \sigma)^{\text{an}})$ and of $\text{coker}(\mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau) \hookrightarrow (\text{Ind}_P^G \sigma)^{\text{an}})$ is not of this form. This is clear for the cokernel (by Corollary 2.34) because $L(\sigma'_0)$ occurs with multiplicity one in $M(\sigma'_0)$. For the kernel, note first that $V' \in \mathcal{O}^G$ implies that it is a finite-dimensional locally analytic representation of P on which \mathfrak{l}_C acts as a direct sum of absolutely simple \mathfrak{l}_C -modules. Hence $(\text{Ind}_P^G V \otimes \tau)^{\text{an}} \cong \mathcal{F}_P^G(\underline{M}(V'|_P), \tau)$ by Proposition 2.17, so by Corollary 2.34 it suffices to show that

$$(2.4) \quad [\underline{M}(V'/\sigma'_0) : L(\sigma'_0)]_{\mathcal{O}^P} = 0.$$

Write $\sigma'_0 \cong L_L(\lambda)$ as \mathfrak{l}_C -module for some $\lambda \in (\mathfrak{t}')^*$, so $V' \cong L(\lambda)$ as \mathfrak{g}_C -module. Thus only weights $< \lambda$ occur in V'/σ'_0 , which implies (2.4).

Let $\pi \subset \mathcal{F}_P^G(\underline{L}(\sigma'_0), \tau) \subset (\text{Ind}_P^G \sigma)^{\text{an}}$ be an irreducible subrepresentation and take an irreducible subrepresentation π_0 of $(\text{Ind}_P^G \tau)^{\text{sm}}$ such that π is the image of $V \otimes \pi_0 \subset V \otimes (\text{Ind}_P^G \tau)^{\text{sm}}$. Then π_0 is dense in $(\text{Ind}_P^G \tau)^{\text{cts}}$ by assumption, so also $V \otimes \pi_0$ is dense in $V \otimes (\text{Ind}_P^G \tau)^{\text{cts}}$. The above diagram then shows that π is dense in $(\text{Ind}_P^G \sigma)^{\text{cts}}$. \square

Theorem 2.53. *Assume Assumption 2.14. The Banach space representation $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible if and only if condition $(*)$ above holds.*

Proof. Suppose that $(*)$ holds and that $\pi \subset \text{Ind}_P^G \sigma$ is a nonzero closed subrepresentation. Then $\pi^{\text{an}} \neq 0$ by Theorem 2.1. We have $(\text{Ind}_P^G \sigma)^{\text{an}} \cong \mathcal{F}_P^G(\underline{M}(\sigma'_0), \tau)$. By Corollary 2.31 we have

$$\text{soc}_G((\text{Ind}_P^G \sigma)^{\text{an}}) \cong \mathcal{F}_Q^G(\underline{L}(\sigma'_0), \text{soc}_{L_Q}(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}),$$

and hence π^{an} contains $\mathcal{F}_Q^G(\underline{L}(\sigma'_0), \rho)$ for some irreducible subrepresentation ρ of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$. Let $\tilde{\rho} := \underline{L}_{L_Q}(\sigma'_0)' \otimes \rho$ denote the corresponding (topologically) irreducible subrepresentation of $\mathcal{F}_{P \cap L_Q}^{L_Q}(\underline{L}_{L_Q}(\sigma'_0), \tau)$. Note that $\underline{L}(\underline{L}_{L_Q}(\sigma'_0)) \cong \underline{L}(\sigma'_0)$ by Corollary 2.12. Therefore $\mathcal{F}_Q^G(\underline{L}(\sigma'_0), \rho)$ is contained in

$$(2.5) \quad \begin{aligned} \mathcal{F}_Q^G(\underline{M}(\underline{L}_{L_Q}(\sigma'_0)), \rho) &\cong (\text{Ind}_Q^G \underline{L}_{L_Q}(\sigma'_0)' \otimes \rho)^{\text{an}} = (\text{Ind}_Q^G \tilde{\rho})^{\text{an}} \\ &\subset (\text{Ind}_Q^G (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{an}})^{\text{an}} \cong (\text{Ind}_P^G \sigma)^{\text{an}}. \end{aligned}$$

By Proposition 2.36 (and Lemma 2.11) we deduce that π^{an} contains all functions in the right-hand side of (2.5) that are supported on $\overline{N}_{Q,0}Q/Q$ and are locally constant on $\overline{N}_{Q,0}$ (where $\overline{N}_{Q,0}$ is a fixed compact open subgroup of \overline{N}_Q). This space is isomorphic to $C^\infty(\overline{N}_{Q,0}, \tilde{\rho})$. By condition $(*)$ we know that $\tilde{\rho}$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}$, and hence $C^\infty(\overline{N}_{Q,0}, \tilde{\rho})$ is dense in $C^0(\overline{N}_{Q,0}, (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}})$. But $C^0(\overline{N}_{Q,0}, (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}})$ generates $(\text{Ind}_P^G \sigma)^{\text{cts}} \cong (\text{Ind}_Q^G (\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}})^{\text{cts}}$ as G -representation.

Conversely, if $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible, then $(\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}$ is irreducible (as follows from the exactness of $\text{Ind}_Q^G(-) \cong C^0(G/Q, -)$, cf. [Eme17, (2.1.3)] and [BH20, Corollary 2.2]), hence condition $(*)$ holds. \square

Corollary 2.54. *Assume Assumption 2.14. If every irreducible subrepresentation of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is dense in $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$, then $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible. In particular, if $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is irreducible, then $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible.*

Proof. Note that $\mathcal{F}_{P \cap L_Q}^{L_Q}(1, \tau) = (\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$. Hence the corollary follows from Lemma 2.52 and Theorem 2.53. The last statement follows from the fact that $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is dense in $(\text{Ind}_P^G \tau)^{\text{cts}}$. This follows, for example, from Lemma 2.2. \square

Corollary 2.55. *Assume Assumption 2.14. Suppose that every irreducible subrepresentation of $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is absolutely irreducible (as is true after some finite extension of C). Then condition $(*)$ implies that $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is absolutely irreducible.*

Proof. We just need to note that Ind_P^G and soc_G commute with finite scalar extensions. \square

Theorem 2.56. *Assume Assumption 2.14. The following are equivalent:*

- (i) $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible;
- (ii) $(\text{Ind}_{P \cap L_Q}^{L_Q} \sigma)^{\text{cts}}$ is irreducible;
- (iii) $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{cts}}$ is irreducible.

In particular, to understand when $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible it suffices to restrict to the case where σ is smooth!

Proof. Recall that $\underline{L}_{L_Q}(W) \in \mathcal{O}^{L_Q}$ (i.e. is equimaximal) by Corollary 2.12, so parts (i) and (ii) are equivalent because condition $(*)$ is literally the same in both cases. We know that (iii) implies (i) by Lemma 2.52 and Theorem 2.53. To prove that (ii) implies (iii), we relabel L_Q as G and may therefore assume that $\underline{L}(\sigma'_0) \in \mathcal{O}^G$ is a finite-dimensional locally analytic representation of G . Let $V := \underline{L}(\sigma'_0)'$. Take any nonzero closed subrepresentation $\pi \subset (\text{Ind}_P^G \tau)^{\text{cts}}$. We first claim that the composition (already considered in the proof of Lemma 2.52)

$$(2.6) \quad V \otimes \pi^{\text{an}} \hookrightarrow V \otimes (\text{Ind}_P^G \tau)^{\text{an}} \xrightarrow{\sim} (\text{Ind}_P^G V \otimes \tau)^{\text{an}} \twoheadrightarrow (\text{Ind}_P^G \sigma)^{\text{an}}$$

is surjective.

To prove the claim, by Lemma 2.3 the maps (2.6) are obtained by applying the functor of locally \mathbb{Q}_p -analytic vectors to the maps

$$(2.7) \quad V \otimes \pi \hookrightarrow V \otimes (\text{Ind}_P^G \tau)^{\text{cts}} \xrightarrow{\sim} (\text{Ind}_P^G V \otimes \tau)^{\text{cts}} \twoheadrightarrow (\text{Ind}_P^G \sigma)^{\text{cts}}$$

of admissible Banach space representations of G . The surjectivity of the composition (2.6) is equivalent to the surjectivity of the composition (2.7). (Note that the image of the composition (2.7) is closed by admissibility and that the functor $(\cdot)^{\text{an}}$ is exact.) By assumption, $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible, so it suffices to show that the composition (2.7) is nonzero or equivalently that the composition (2.6) is nonzero. By Corollary 2.31 applied with $M = 1$ (the trivial representation), $Q = G$ and $\pi = \tau$ we see that any nonzero closed subrepresentation of $(\text{Ind}_P^G \tau)^{\text{an}}$ intersects $(\text{Ind}_P^G \tau)^{\text{sm}}$ non-trivially. But the composition

$$V \otimes (\text{Ind}_P^G \tau)^{\text{sm}} \hookrightarrow V \otimes (\text{Ind}_P^G \tau)^{\text{an}} \xrightarrow{\sim} (\text{Ind}_P^G V \otimes \tau)^{\text{an}} \twoheadrightarrow (\text{Ind}_P^G \sigma)^{\text{an}}$$

is injective by the proof of Lemma 2.52, hence indeed the claim holds.

Recall that we have fixed a system of positive roots for (\mathfrak{g}_C, ι') , and let λ be the lowest weight of V . Applying the projection functor $\text{pr}_{\lambda-\rho}$ to (2.6), we get

$$(2.8) \quad \text{pr}_{\lambda-\rho}(V \otimes \pi^{\text{an}}) \hookrightarrow \text{pr}_{\lambda-\rho}(V \otimes (\text{Ind}_P^G \tau)^{\text{an}}) \xrightarrow{\sim} \text{pr}_{\lambda-\rho}((\text{Ind}_P^G V \otimes \tau)^{\text{an}}) \twoheadrightarrow (\text{Ind}_P^G \sigma)^{\text{an}}.$$

By the definition of translation functors we have $\text{pr}_{\lambda-\rho}(V \otimes \pi^{\text{an}}) = T_{-\rho}^{\lambda-\rho}(G, V)(\pi^{\text{an}})$ and $\text{pr}_{\lambda-\rho}(V \otimes (\text{Ind}_P^G \tau)^{\text{an}}) = T_{-\rho}^{\lambda-\rho}(G, V)((\text{Ind}_P^G \tau)^{\text{an}})$. By Proposition 2.48(ii), the last map in (2.8) is an isomorphism. Hence the map

$$T_{-\rho}^{\lambda-\rho}(G, V)(\pi^{\text{an}}) \twoheadrightarrow T_{-\rho}^{\lambda-\rho}((\text{Ind}_P^G \tau)^{\text{an}})$$

is surjective. Apply $T_{\lambda-\rho}^{-\rho}(G, V')$. By Proposition 2.47, the inclusion $\pi^{\text{an}} \rightarrow (\text{Ind}_P^G \tau)^{\text{an}}$ is surjective. Therefore $\pi = (\text{Ind}_P^G \tau)^{\text{cts}}$ (after taking closure), hence we get (iii). \square

Corollary 2.57. *Assume Assumption 2.14. We continue to write $\sigma = \sigma_0 \otimes \tau$ as above, with $Q \supset P$ denoting the maximal parabolic of $\underline{L}(\sigma'_0)$. Each of the following conditions implies the next:*

- (i) $M(\sigma'_0) = U(\mathfrak{g}_C) \otimes_{U(\mathfrak{p}_C)} \sigma'_0$ is irreducible as $U(\mathfrak{g}_C)$ -module;
- (ii) $Q = P$;
- (iii) $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is irreducible.

Proof. If (i) holds, and Q is strictly bigger than P , then $L(\sigma'_0)^{\mathfrak{n}_{Q,C}} \in \mathcal{O}^{\mathfrak{l}_Q}$ is finite-dimensional, so the natural map $U(\mathfrak{l}_{Q,C}) \otimes_{U(\mathfrak{p}_C \cap \mathfrak{l}_{Q,C})} \sigma'_0 \rightarrow L(\sigma'_0)^{\mathfrak{n}_{Q,C}}$ is not an isomorphism, so after extending scalars to $U(\mathfrak{g}_C)$ we obtain

$$M(\sigma'_0) \twoheadrightarrow U(\mathfrak{g}_C) \otimes_{U(\mathfrak{q}_C)} L(\sigma'_0)^{\mathfrak{n}_{Q,C}} \twoheadrightarrow L(\sigma'_0),$$

where the first map is not an isomorphism, contradicting (i).

If (ii) holds, then we deduce (iii) from Corollary 2.54. \square

Remark 2.58. Note that part (i) (when $\dim_C \tau = 1$) is Orlik–Strauch’s irreducibility criterion for the locally analytic representation $(\text{Ind}_P^G \sigma)^{\text{an}}$ [OS10], which in turn implies (iii). We remark that this criterion also follows from Theorem 2.16. The application of this criterion to the irreducibility of Banach representation was noted in [Sch11, Proposition 2.6(ii)].

Proposition 2.59. *Assume Assumption 2.14. Suppose that $\underline{G}_1, \underline{G}_2$ are connected reductive groups, $\underline{P}_1 = \underline{L}_1 \underline{N}_1 \subset \underline{G}_1$, $\underline{P}_2 = \underline{L}_2 \underline{N}_2 \subset \underline{G}_2$ parabolic subgroups and σ_1 (resp. σ_2) a finite-dimensional absolutely irreducible continuous representation of \underline{L}_1 (resp. \underline{L}_2). Then $(\text{Ind}_{\underline{P}_1 \times \underline{P}_2}^{\underline{G}_1 \times \underline{G}_2} \sigma_1 \boxtimes \sigma_2)^{\text{cts}}$ is absolutely irreducible if and only if $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \sigma_1)^{\text{cts}}$ and $(\text{Ind}_{\underline{P}_2}^{\underline{G}_2} \sigma_2)^{\text{cts}}$ are both absolutely irreducible.*

Proof. We first note that $(\text{Ind}_{\underline{P}_1}^{\underline{G}_1} \sigma_1)^{\text{cts}} \widehat{\boxtimes} (\text{Ind}_{\underline{P}_2}^{\underline{G}_2} \sigma_2)^{\text{cts}} \cong (\text{Ind}_{\underline{P}_1 \times \underline{P}_2}^{\underline{G}_1 \times \underline{G}_2} \sigma_1 \boxtimes \sigma_2)^{\text{cts}}$ as admissible Banach representations of $\underline{G}_1 \times \underline{G}_2$. (This follows exactly as in the proof of [BH20, Lemma 2.8], using the isomorphism $C^0(P_1 \backslash G_1, \sigma_1) \widehat{\otimes} C^0(P_2 \backslash G_2, \sigma_2) \cong C^0(P_1 \backslash G_1 \times P_2 \backslash G_2, \sigma_1 \otimes \sigma_2)$, cf. the end of [Sch02, §17], instead of [BH20, (2.6)].) The “only if” direction follows (using, for example, [BH20, Lemma 2.1(ii)]).

For the “if” direction, by a z -extension we may assume $\underline{G}_1^{\text{der}}, \underline{G}_2^{\text{der}}$ are simply connected. Then by Lemma 2.51, after perhaps replacing C by a finite extension, we can take a decomposition $\sigma_i = \sigma_{i,0} \otimes \tau_i$ as in this subsection, for $i = 1, 2$. Then $\sigma_1 \boxtimes \sigma_2 \cong (\sigma_{1,0} \boxtimes \sigma_{2,0}) \otimes (\tau_1 \boxtimes \tau_2)$, where $\sigma_{1,0} \boxtimes \sigma_{2,0}$ is $\mathfrak{l}_{1,C} \times \mathfrak{l}_{2,C}$ -simple and $\underline{L}(\sigma'_{1,0} \boxtimes \sigma'_{2,0}) \cong \underline{L}(\sigma'_{1,0}) \boxtimes \underline{L}(\sigma'_{2,0})$ equimaximal with maximal parabolic $Q_1 \times Q_2$. Let $P(Q_i) := P_i \cap L_{Q_i}$ for $i = 1, 2$. By Corollary 2.54, it suffices to show that every irreducible subrepresentation of $(\text{Ind}_{P(Q_1) \times P(Q_2)}^{L_{Q_1} \times L_{Q_2}} \tau_1 \boxtimes \tau_2)^{\text{sm}}$ is dense in $(\text{Ind}_{P(Q_1) \times P(Q_2)}^{L_{Q_1} \times L_{Q_2}} \tau_1 \boxtimes \tau_2)^{\text{cts}}$. An irreducible subrepresentation of $(\text{Ind}_{P(Q_1) \times P(Q_2)}^{L_{Q_1} \times L_{Q_2}} \tau_1 \boxtimes \tau_2)^{\text{sm}} \cong (\text{Ind}_{P(Q_1)}^{L_{Q_1}} \tau_1)^{\text{sm}} \boxtimes (\text{Ind}_{P(Q_2)}^{L_{Q_2}} \tau_2)^{\text{sm}}$ is of the form $\pi_1 \boxtimes \pi_2$ where $\pi_i \subset (\text{Ind}_{P(Q_i)}^{L_{Q_i}} \tau_i)^{\text{sm}}$ is an irreducible subrepresentation for $i = 1, 2$. By assumption and Theorem 2.56, π_i is dense in $(\text{Ind}_{P(Q_i)}^{L_{Q_i}} \tau_i)^{\text{cts}}$ for $i = 1, 2$. Hence $\pi_1 \boxtimes \pi_2 \subset (\text{Ind}_{P(Q_1)}^{L_{Q_1}} \tau_1)^{\text{cts}} \widehat{\boxtimes} (\text{Ind}_{P(Q_2)}^{L_{Q_2}} \tau_2)^{\text{cts}} \cong (\text{Ind}_{P(Q_1) \times P(Q_2)}^{L_{Q_1} \times L_{Q_2}} \tau_1 \boxtimes \tau_2)^{\text{cts}}$ is dense, as required. \square

Remark 2.60. The problem of understanding when a parabolic induction $(\text{Ind}_P^G \sigma)^{\text{cts}}$ (with σ finite-dimensional) is absolutely irreducible can be reduced to the case where \underline{G} is an absolutely almost simple simply-connected group. (First apply Remark 2.44 to reduce to the simply-connected cover of $\underline{G}^{\text{der}}$. Then use Proposition 2.59 to reduce to an almost simple simply-connected group. Then observe that $\underline{G} = \text{Res}_{E/F} \underline{H}$ with \underline{H} absolutely almost simple simply-connected.) We may moreover assume that \underline{G} is isotropic, as otherwise \underline{G} is the only parabolic subgroup.

2.9. Genericity. Recall that $\underline{B} = \underline{Z}\underline{U}$ is a minimal parabolic subgroup. We assume that we are in the setting of §2.8 with $\underline{P} = \underline{B}$, i.e. $F = \mathbb{Q}_p$ and $\sigma = \sigma_0 \otimes \tau$, where $\sigma_0 \in \mathcal{O}^Z$ is \mathfrak{z}_C -simple and τ is an absolutely irreducible smooth Z -representation such that moreover $\underline{L}(\sigma'_0) \in \mathcal{O}^B$ is equimaximal with maximal parabolic $Q = L_Q N_Q$.

A smooth character $\theta: U \rightarrow C^\times$ is called *non-degenerate* if the restriction of θ to each simple root subgroup is non-trivial. For a smooth representation π of U , let $\pi_{U,\theta} := \pi / \langle \pi(u)v - \theta(u)v \mid u \in U \rangle$ be the space of twisted coinvariants. We say that a representation π of G is *generic* if $\pi_{U,\theta} \neq 0$ for some non-degenerate θ . When \underline{G} is quasisplit, this is the familiar notion.

Proposition 2.61. *Assume Assumption 2.14. If any irreducible subrepresentation of $(\text{Ind}_{B \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is generic, then $(\text{Ind}_B^G \sigma)^{\text{cts}}$ is irreducible.*

Proof. To simplify notation, we relabel L_Q as G . Let W be the Weyl group of G and $w_0 \in W$ the longest element. Then we have the Bruhat decomposition $G = \coprod_{w \in W} BwB$ and $(\text{Ind}_B^G \tau)^{\text{sm}}$ has a B -stable filtration F_w with graded pieces $(\text{c-Ind}_B^{BwB} \tau)^{\text{sm}}$ for $w \in W$, and $F_{w_1} \subset F_{w_2}$ if $w_1 \geq w_2$ with respect to the Bruhat order.

Let $\pi \subset (\text{Ind}_B^G \tau)^{\text{sm}}$ be an irreducible subrepresentation and take a non-degenerate character $\theta: U \rightarrow C^\times$ such that $\pi_{U,\theta} \neq 0$. From an argument in [CS80, p. 211] or the proof of the geometric lemma [BZ77, 5.2 Theorem] we have $((\text{c-Ind}_B^{BwB} \tau)^{\text{sm}})_{U,\theta} = 0$ if $w \neq w_0$.

By above, and as $(-)_U$ is an exact functor on the category of smooth U -representations, we have an exact sequence of smooth B -representations

$$0 \rightarrow (\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}} \rightarrow (\text{Ind}_B^G \tau)^{\text{sm}} \rightarrow \pi_0 \rightarrow 0$$

with $(\pi_0)_{U,\theta} = 0$, which induces an exact sequence

$$(2.9) \quad 0 \rightarrow \pi \cap (\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}} \rightarrow \pi \rightarrow \pi_1 \rightarrow 0$$

with $(\pi_1)_{U,\theta} = 0$. By assumption we have $\pi_{U,\theta} \neq 0$ and hence by (2.9) we have $(\pi \cap (\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}})_{U,\theta} \neq 0$. Note that the map $(\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}} \rightarrow \tau$, $f \mapsto \int_U f(w_0u)\theta^{-1}(u)du$ identifies τ with the twisted coinvariants of $(\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}}$. We can thus take $f \in \pi \cap (\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}}$ such that $v := \int_U f(w_0u)\theta^{-1}(u)du \neq 0$. Suppose that $\text{supp}(f) \subset B \backslash Bw_0U_0$ for some compact open subgroup U_0 of U . Then $f' := \int_{U_0} (u_0f) \cdot \theta^{-1}(u_0)du_0 \in \pi \cap (\text{c-Ind}_B^{Bw_0B} \tau)^{\text{sm}}$ is supported on $B \backslash Bw_0U_0$ and $f'(w_0u_0) = \theta(u_0)v \in Cv$. By Lemma 2.2 we see that π is dense in $(\text{Ind}_B^G \tau)^{\text{cts}}$ and we get the proposition by Corollary 2.54. \square

3. APPLICATIONS

In this section, we give applications of our irreducibility criterion.

Recall that we have fixed a maximal split torus \underline{S} of \underline{G} and a minimal parabolic subgroup $\underline{B} = \underline{Z}\underline{U}$ such that $\underline{S} \subset \underline{Z}$. Let $\underline{P} = \underline{L}\underline{N}$ be a parabolic subgroup, $\underline{\overline{P}} = \underline{\overline{L}}\underline{\overline{N}}$ the opposite parabolic subgroup. Let $\underline{A}_{\underline{L}} \subset \underline{L}$ be the maximal split torus in the center of \underline{L} and $\Phi(\underline{G}, \underline{A}_{\underline{L}})$ the set of roots of $\underline{A}_{\underline{L}}$. We have $\underline{A}_{\underline{L}} \subset \underline{A}_{\underline{Z}}$ and $\Phi(\underline{G}, \underline{A}_{\underline{L}}) = \{\alpha|_{\underline{A}_{\underline{L}}} \mid \alpha \in \Phi(\underline{G}, \underline{A}_{\underline{Z}})\} \setminus \{0\}$. Let $\Phi(\underline{P}, \underline{A}_{\underline{L}})$ be the set of $\alpha \in \Phi(\underline{G}, \underline{A}_{\underline{L}})$ that appear in $\text{Lie}(\underline{N})$. For $\alpha \in \Phi(\underline{G}, \underline{A}_{\underline{L}})$, let \underline{L}_α be the centralizer of the connected component of $\ker \alpha \subset \underline{A}_{\underline{L}}$ in \underline{G} . It is a Levi subgroup containing \underline{L} and $\underline{P} \cap \underline{L}_\alpha$ is a maximal parabolic subgroup of \underline{L}_α . Let $\Phi_{\text{red}}(\underline{P}, \underline{A}_{\underline{L}})$ (resp. $\Phi_{\text{red}}(\underline{G}, \underline{A}_{\underline{L}})$) be the set of reduced elements in $\Phi(\underline{P}, \underline{A}_{\underline{L}})$ (resp. $\Phi(\underline{G}, \underline{A}_{\underline{L}})$) and $\Delta(\underline{P}, \underline{A}_{\underline{L}})$ the set of simple roots in $\Phi(\underline{P}, \underline{A}_{\underline{L}})$. When \underline{P} is a minimal parabolic subgroup $\underline{B} = \underline{Z}\underline{U}$, we put $\Phi := \Phi(\underline{G}, \underline{A}_{\underline{Z}})$, $\Phi_{\text{red}} := \Phi_{\text{red}}(\underline{G}, \underline{A}_{\underline{Z}})$, $\Phi_{\text{red}}^+ := \Phi_{\text{red}}(\underline{B}, \underline{A}_{\underline{Z}})$, $\Delta := \Delta(\underline{B}, \underline{A}_{\underline{Z}})$. If \underline{P} is standard, then $\Delta(\underline{P}, \underline{A}_{\underline{L}}) = \{\alpha|_{\underline{A}_{\underline{L}}} \mid \alpha \in \Delta\} \setminus \{0\}$. In general, let $X^*(\underline{H})$ (resp. $X_*(\underline{H})$) be the rational character (resp. cocharacter) group of an algebraic group \underline{H} . For $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ we let $\mathfrak{a}_{L,K}^* := X^*(\underline{A}_{\underline{L}}) \otimes K = X^*(\underline{L}) \otimes K$ and denote by $\mathfrak{a}_{L,K}$ its dual vector space.

If $\underline{L}_1 \subset \underline{L}_2$ are semistandard Levi subgroups we have $\underline{A}_{\underline{L}_2} \subset \underline{A}_{\underline{L}_1}$ and hence get canonical maps $\mathfrak{a}_{L_1,\mathbb{Q}}^* \rightarrow \mathfrak{a}_{L_2,\mathbb{Q}}^*$ and $\mathfrak{a}_{L_2,\mathbb{Q}}^* \hookrightarrow \mathfrak{a}_{L_1,\mathbb{Q}}^*$ giving a splitting $\mathfrak{a}_{L_1,\mathbb{Q}}^* = \mathfrak{a}_{L_2,\mathbb{Q}}^* \oplus (\mathfrak{a}_{L_1,\mathbb{Q}}^{L_2})^*$ (and similarly over \mathbb{R} and \mathbb{C}). We note that if $\underline{P} = \underline{L}\underline{N}$ is semistandard, then $\Delta(\underline{P}, \underline{A}_{\underline{L}})$ is a \mathbb{Q} -basis of $(\mathfrak{a}_{L,\mathbb{Q}}^G)^*$.

Given $\alpha \in \Phi_{\text{red}}(\underline{G}, \underline{A}_{\underline{L}})$ we define the coroot $\alpha^\vee \in \mathfrak{a}_{L,\mathbb{Q}}$ as follows. The subspace $(\mathfrak{a}_{L,\mathbb{Q}}^{L_\alpha})^*$ is one-dimensional with basis α and we let α^\vee be the unique element of $\mathfrak{a}_{L,\mathbb{Q}}^{L_\alpha}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. Note that if $L = Z$, then this coincides with the usual notion of relative coroot $(\alpha^\vee \in X_*(\underline{A}_{\underline{Z}}) \subset \mathfrak{a}_{Z,\mathbb{Q}})$.

It is convenient to fix a W -invariant positive definite inner product on $\mathfrak{a}_{Z,\mathbb{R}}^*$. Then any decomposition $\mathfrak{a}_{L,\mathbb{R}}^* = \mathfrak{a}_{G,\mathbb{R}}^* \oplus (\mathfrak{a}_{L,\mathbb{R}}^G)^*$ is orthogonal (checking first for the pairs (Z, G) and (Z, L)). Hence the isomorphism $\mathfrak{a}_{Z,\mathbb{R}} \xrightarrow{\sim} \mathfrak{a}_{Z,\mathbb{R}}^*$ defined by the inner product identifies $\mathfrak{a}_{L,\mathbb{R}}$ with $\mathfrak{a}_{L,\mathbb{R}}^*$ and $\mathfrak{a}_{L,\mathbb{R}}^G$ with $(\mathfrak{a}_{L,\mathbb{R}}^G)^*$. In particular, taking $G = L_\alpha$ we see that $\alpha^\vee \in \mathfrak{a}_{L,\mathbb{R}}$ is identified with $2\alpha/(\alpha, \alpha) \in \mathfrak{a}_{L,\mathbb{R}}^*$.

Lemma 3.1. *Suppose $\underline{L}_1 \subset \underline{L}$ are semistandard Levi subgroups.*

- (i) *Take $\alpha \in \Phi_{\text{red}}(\underline{G}, \underline{A}_{\underline{L}_1})$ such that $\alpha_L = \alpha|_{\underline{A}_{\underline{L}}} \neq 0$. Then the image of α^\vee under $\mathfrak{a}_{L_1,\mathbb{Q}} \rightarrow \mathfrak{a}_{L,\mathbb{Q}}$ lies in $\mathbb{Q}_{>0}\alpha_L^\vee$.*
- (ii) *If $\alpha_1, \dots, \alpha_\ell \in \Phi_{\text{red}}(\underline{G}, \underline{A}_{\underline{L}})$ are linearly independent in $\mathfrak{a}_{L_1,\mathbb{R}}^*$, then $\alpha_1^\vee, \dots, \alpha_\ell^\vee$ are linearly independent in $\mathfrak{a}_{L_1,\mathbb{R}}$.*

Proof. (i) The chosen inner product identifies the projection $\mathfrak{a}_{L_1,\mathbb{R}} \rightarrow \mathfrak{a}_{L,\mathbb{R}}$ with the projection $\mathfrak{a}_{L_1,\mathbb{R}}^* \rightarrow \mathfrak{a}_{L,\mathbb{R}}^*$. Hence the projection of α^\vee equals $\frac{(\alpha_L, \alpha_L)}{(\alpha, \alpha)}\alpha_L^\vee$.

(ii) This is obvious by identifying $\mathfrak{a}_{L_1,\mathbb{R}}$ and $\mathfrak{a}_{L_1,\mathbb{R}}^*$ via the chosen inner product. \square

Lemma 3.2. *Let $\underline{P} = \underline{L}\underline{N}$ be a standard parabolic subgroup. Write $\Delta_L := \Delta(\underline{B} \cap \underline{L}, \underline{A}_{\underline{L}})$.*

- (i) *If $\alpha \in \Delta \setminus \Delta_L$, then $\alpha^\vee \in \mathbb{Q}_{>0}\alpha_L^\vee \oplus \mathbb{Q}_{\leq 0}\Delta_L^\vee$, where $\alpha_L := \alpha|_{\underline{A}_{\underline{L}}} \neq 0$.*
- (ii) *Suppose that $x \in \mathfrak{a}_{Z,\mathbb{R}}^*$. Write $x = x_L + x'_L$ with $x_L \in \mathfrak{a}_{L,\mathbb{R}}^*$ and $x'_L \in (\mathfrak{a}_{Z,\mathbb{R}}^L)^*$. If x is dominant, then x_L is dominant.*

Here, $x_L \in \mathfrak{a}_{L,\mathbb{R}}^*$ is *dominant* if $\langle x_L, \alpha_L^\vee \rangle \geq 0$ for all $\alpha_L \in \Phi(\underline{P}, \underline{A}_{\underline{L}})$, or equivalently for all $\alpha_L \in \Delta(\underline{P}, \underline{A}_{\underline{L}})$. By Lemma 3.1 this is equivalent to x_L being dominant as an element in $\mathfrak{a}_{Z,\mathbb{R}}^*$.

Proof. (i) Write $\alpha^\vee = x + y$ with $x \in \mathfrak{a}_{L,\mathbb{Q}}$ and $y = \sum_{\beta \in \Delta_L} \lambda_\beta \beta^\vee \in \mathfrak{a}_{Z,\mathbb{Q}}^L$ with $\lambda_\beta \in \mathbb{Q}$. By Lemma 3.1(i) we have $x = \lambda\alpha_L^\vee$ with $\lambda \in \mathbb{Q}_{>0}$. On the other hand, for all $\gamma \in \Delta_L$ we get $0 \geq \langle \gamma, \alpha^\vee \rangle = \sum_{\beta \in \Delta_L} \lambda_\beta \langle \gamma, \beta^\vee \rangle$. The fundamental coweight ϖ_δ of $\delta \in \Delta_L$ in $(\mathfrak{a}_{Z,\mathbb{R}}^L)^*$ is contained in $\sum_{\gamma \in \Delta_L} \mathbb{R}_{\geq 0}\gamma$, whence $0 \geq \lambda_\delta$ for all $\delta \in \Delta_L$.

(ii) If $\alpha \in \Delta_L$, then $\alpha^\vee \in \Delta_L^\vee \subset \mathfrak{a}_{Z,\mathbb{R}}^L$, so $\langle x'_L, \alpha^\vee \rangle = \langle x, \alpha^\vee \rangle \geq 0$. If $\alpha \in \Delta \setminus \Delta_L$, write $\alpha^\vee = \lambda\alpha_L^\vee + \sum_{\beta \in \Delta_L} \lambda_\beta \beta^\vee$ by (i), with $\lambda > 0$ and $\lambda_\beta \leq 0$. Then $\lambda \langle x_L, \alpha_L^\vee \rangle = \langle x, \alpha^\vee \rangle - \sum_{\beta \in \Delta_L} \lambda_\beta \langle x'_L, \beta^\vee \rangle \geq 0$, so $\langle x_L, \alpha_L^\vee \rangle \geq 0$. \square

3.1. On reducibility points of parabolic induction. Let $\underline{P} = \underline{L}\underline{N}$ be a parabolic subgroup. By Corollary 2.54, to prove $(\text{Ind}_{\underline{P}}^G \sigma)^{\text{cts}}$ is irreducible, it is sufficient to prove that $(\text{Ind}_{\underline{P} \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is irreducible for a certain parabolic subgroup \underline{Q} containing \underline{P} , at least whenever σ can be decomposed as tensor product $\sigma_0 \otimes \tau$ as in subsection 2.8 (for example, when $\underline{G}^{\text{der}}$ is simply connected). We collect some known facts about the reducibility of smooth parabolic inductions over \mathbb{C} .

Let σ_0 be an irreducible smooth complex representation of L and $\mathcal{O}_{\mathbb{C}}$ be the set of isomorphism classes of $\sigma_0 \otimes \chi$, where $\chi: L \rightarrow \mathbb{C}^\times$ is an unramified character. We have Harish-Chandra's homomorphism $H_L: L \rightarrow \mathfrak{a}_{L,\mathbb{R}}$ normalized by $q^{\langle \chi, H_L(\ell) \rangle} = |\chi(\ell)|_F$ for all $\chi \in X^*(\underline{L})$ and $\ell \in L$. Then for $\nu \in \mathfrak{a}_{L,\mathbb{C}}^*$ we define the unramified character $\chi_\nu: L \rightarrow \mathbb{C}^\times$ by $\chi_\nu(\ell) := q^{\langle \nu, H_L(\ell) \rangle}$. Then the map $\nu \mapsto \chi_\nu$ identifies the group of unramified characters $X_{\text{nr}}(L)$ with the quotient of $\mathfrak{a}_{L,\mathbb{C}}^*$ by a lattice in $i\mathfrak{a}_{L,\mathbb{R}}^*$ (a complex torus with character group $L/\ker H_L$). In this way $\mathcal{O}_{\mathbb{C}}$ has the structure of an algebraic variety over \mathbb{C} (a homogeneous space for $X_{\text{nr}}(L)$ with finite stabilizer subgroups).

Let $\underline{Q} = \underline{L}\underline{N}_{\underline{Q}}$ be a semistandard parabolic subgroup which has the same Levi part as \underline{P} . For $\sigma \in \mathcal{O}_{\mathbb{C}}$, we have an intertwining operator $J_{\underline{Q}|\underline{P}}(\sigma): (\text{n-Ind}_{\underline{P}}^G \sigma)^{\text{sm}} \rightarrow$

$(\mathfrak{n}\text{-Ind}_Q^G \sigma)^{\text{sm}}$ defined by

$$(J_{Q|P}(\sigma)f)(g) = \int_{(N \cap N_Q) \backslash N_Q} f(ng) dn.$$

It converges if (the unramified part of) σ is sufficiently dominant, and has meromorphic continuation to $\mathcal{O}_{\mathbb{C}}$. In fact it is a rational function on $\mathcal{O}_{\mathbb{C}}$, see [Wal03, Théorème IV.1.1].

The definition of $J_{Q|P}$ depends on a choice of Haar measure. Here we fix a measure as follows: we have a bijective map $\prod_{\alpha \in \Phi_{\text{red}}(\underline{Q}, \underline{A}_L) \setminus \Phi_{\text{red}}(\underline{P}, \underline{A}_L)} (N_Q \cap L_\alpha) \xrightarrow{\sim} (N \cap N_Q) \backslash N_Q$, where we fix an order of $\Phi_{\text{red}}(\underline{Q}, \underline{A}_L) \setminus \Phi_{\text{red}}(\underline{P}, \underline{A}_L)$. For each $\alpha \in \Phi_{\text{red}}(\underline{Q}, \underline{A}_L) \setminus \Phi_{\text{red}}(\underline{P}, \underline{A}_L)$ we fix a Haar measure on $N_Q \cap L_\alpha$ and take the product measure on $(N \cap N_Q) \backslash N_Q$. We will fix more specific measures on $N_Q \cap L_\alpha$ later.

The most important case is when σ_0 is a *discrete series* and $\underline{Q} = \overline{P}$. The set of $\sigma \in \mathcal{O}_{\mathbb{C}}$ such that $(\mathfrak{n}\text{-Ind}_P^G \sigma)^{\text{sm}}$ is irreducible is open and non-empty [Wal03, Proposition IV.2.2]. Hence there exists a rational function $j(\sigma)$ such that $J_{P|\overline{P}}(\sigma) J_{\overline{P}|P}(\sigma) = j(\sigma)$. Note that $j(\sigma)$ does not depend on P [Wal03, IV.3(1)]. We define Harish-Chandra's rational function $\mu^G(\sigma)$ by the same formula as in [Wal03, V.2]. (In [Wal03, V.2], σ is assumed to be unitary, however the definition works for any σ and gives a rational function on $\mathcal{O}_{\mathbb{C}}$.) We have $\mu^G(\sigma) \in \mathbb{R}_{>0}^\times \cdot j(\sigma)^{-1}$, where the implied constant only depends on (G, L) . We also have $\mu^G(\sigma) \geq 0$ for all unitary $\sigma \in \mathcal{O}_{\mathbb{C}}$ by [Wal03, Lemme V.2.1]. Finally, it is clear that μ^G is a rational function on the quotient $X_{\text{nr}}(G) \backslash \mathcal{O}_{\mathbb{C}}$. The function $j(\sigma)$ depends on the choice of measure, but $\mu^G(\sigma)$ does not depend on it. The function μ^G gives very precise information about the reducibility points of $(\mathfrak{n}\text{-Ind}_P^G \sigma)^{\text{sm}}$ for σ supercuspidal.

Remark 3.3. We normalize μ^G differently compared to [Sil79]. However, the normalizations agree up to a factor in $\mathbb{R}_{>0}^\times$. This follows from the comparison of [Sil79, Theorem 5.2.4.4] (noting that $\mu(\omega) = \mu(\omega : 0)$ in that reference) and [Wal03, Lemme V.2.2].

Proposition 3.4 (Harish-Chandra's product formula). *If σ is a discrete series we have*

$$\mu^G(\sigma) = \prod_{\alpha \in \Phi_{\text{red}}(\underline{P}, \underline{A}_L)} \mu^{L_\alpha}(\sigma).$$

Proof. This is true for $\sigma \in \mathcal{O}_{\mathbb{C}}$ unitary [Wal03, Lemme V.2.1] and hence for all $\sigma \in \mathcal{O}_{\mathbb{C}}$ since both sides are rational functions. \square

Suppose that P is maximal and σ is a unitary supercuspidal representation of L . Note that the group $N_G(L)/L$ has at most two elements, and let $W_G(\sigma) := \{g \in N_G(L)/L : \sigma \circ \text{Ad}(g) \cong \sigma\}$. Also note that $X_{\text{nr}}(G) \backslash \mathcal{O}_{\mathbb{C}}$ is a torus of rank 1 (without fixed base point). Define $2\rho_P \in X^*(\underline{A}_L)$ (as sum of the roots in $\Phi(\underline{P}, \underline{A}_L)$, with multiplicities) such that $\delta_P = \chi_{2\rho_P}$.

Proposition 3.5. *Keep the above notation.*

- (i) *If $W_G(\sigma) = 1$, then $(\mathfrak{n}\text{-Ind}_P^G \sigma \chi)^{\text{sm}}$ is irreducible for all unramified $\chi : L \rightarrow \mathbb{R}_{>0}^\times$.*
- (ii) *Otherwise, there exists a unique $0 \leq s_0 \leq 1/2$ such that $(\mathfrak{n}\text{-Ind}_P^G \sigma \delta_P^s)^{\text{sm}}$ ($s \in \mathbb{R}$) is reducible if and only if $s \in \{\pm s_0\}$.*
- (iii) *In case (i) and case (ii) when $s_0 = 0$, the function μ^G is holomorphic and non-vanishing at all $\sigma \chi$ with $\chi : L \rightarrow \mathbb{R}_{>0}^\times$ unramified.*
- (iv) *In case (ii) when $s_0 > 0$, the function $\mu^G(\sigma \delta_P^s)$ has a double zero at $s = 0$, simple poles at $s = \pm s_0$, and is holomorphic and non-vanishing at all other $s \in \mathbb{R}$.*

Proof. First suppose that $W_G(\sigma) = 1$. Then (iii) follows from [Sil79, Corollary 5.4.2.2] and [Sil80b, Lemma 1.3]. Then (i) follows from [Sil79, Lemma 5.4.2.4] for χ that do not extend to G . If χ extends to G , by twisting we may suppose $\chi = 1$ and we may suppose that $N_G(L)/L$ has order 2 by [Sil79, Theorem 5.4.4.1]. Then (i) follows by combining Lemmas 5.4.5.2, 5.4.1.5 of [Sil79]. (A different proof of (i) can be found in [Ber92, Theorem 28].)

Now suppose that $W_G(\sigma)$ has order 2. If $\mu^G(\sigma) > 0$, then μ^G is holomorphic and non-vanishing at all $\sigma\chi$ with $\chi: L \rightarrow \mathbb{R}_{>0}^\times$ unramified by [Sil80b, Lemma 1.3]. Otherwise, $\mu^G(\sigma) = 0$. We have a commutative square

$$\begin{array}{ccc} (\mathfrak{a}_{G,\mathbb{C}})^* & \hookrightarrow & (\mathfrak{a}_{L,\mathbb{C}})^* \\ \downarrow & & \downarrow \\ X_{\text{nr}}(G) & \hookrightarrow & X_{\text{nr}}(L) \end{array}$$

where the vertical maps are given by $\nu \mapsto \chi_\nu$. Hence the kernel of the map $(\mathfrak{a}_{L,\mathbb{C}}^G)^* \rightarrow X_{\text{nr}}(G) \setminus \mathcal{O}_{\mathbb{C}}$, $\nu \mapsto \sigma\chi_\nu$ is a lattice $L^*(\sigma)$ (of rank 1) in $\sqrt{-1}(\mathfrak{a}_{L,\mathbb{R}}^G)^*$. There is a unique element $\alpha(\sigma) \in \mathfrak{a}_{L,\mathbb{R}}^G$ such that $q^{\langle \nu, \alpha(\sigma) \rangle} = 1$ if and only if $\nu \in L^*(\sigma)$ and $\langle \rho_P, \alpha(\sigma) \rangle > 0$. We let $z := q^{\langle \nu, \alpha(\sigma) \rangle}$ (a generator of the character lattice of the torus $X_{\text{nr}}(G) \setminus \mathcal{O}_{\mathbb{C}}$). Then $\alpha(\sigma)$ and z agree with the ones defined in [Sil80b], except that our ν becomes $\sqrt{-1}\nu$ in [Sil80b]. From [Sil80b, Theorem 1.6] and $\delta_P^s = \chi_{2s\rho_P}$ we deduce that $\mu^G(\sigma) = 0$ implies that $\mu^G(\sigma\delta_P^s)$ is as described in part (iv), for some $0 < s_0 \leq 1/2$.

Finally, part (ii) follows from parts (iii) and (iv) using [Sil79, Lemma 5.4.2.3] (when $s = 0$) and [Sil79, Lemma 5.4.2.4] (when $s \neq 0$). \square

The following comparison of μ -functions will often be useful, especially in combination with Proposition 3.5.

Proposition 3.6 ([Sol21, Proposition 2.2]). *Let $\varphi: \underline{G}_1 \rightarrow \underline{G}$ be a morphism such that $\varphi(\underline{G}_1^{\text{der}}) = \underline{G}^{\text{der}}$ and $\ker \varphi \subset \underline{Z}_{\underline{G}_1}$. Let σ be a unitary supercuspidal representation of L and σ_1 an irreducible constituent of the inflation $\varphi^*(\sigma)$, a unitary supercuspidal representation of $L_1 := \varphi^{-1}(L)$. Then $\mu^L(\sigma\chi)$ and $\mu^{L_1}(\sigma_1\varphi^*(\chi))$ agree up to nonzero constant as rational functions of $\chi \in X_{\text{nr}}(L)$.*

The following result is crucial to us. Its proof was provided to us by J.-L. Waldspurger.

Proposition 3.7 (Waldspurger). *Assume that \underline{P} is a maximal parabolic subgroup, $\overline{\underline{P}} = \underline{L}\overline{N}$ the parabolic subgroup opposite to \underline{P} and $\langle N, \overline{N} \rangle$ the group generated by N and \overline{N} . Let σ be a unitary supercuspidal representation of L and assume that $(\mathfrak{n}\text{-Ind}_{\underline{P}}^G \sigma \delta_{\underline{P}}^{1/2})^{\text{sm}}$ is reducible. Then σ is trivial on $L \cap \langle N, \overline{N} \rangle$.*

The final statement implies that σ extends to a smooth representation of G that is trivial on N by [AHHV17, II.7 Proposition]. (To see this, choose a minimal parabolic $\underline{B} = \underline{Z}\underline{U} \subset \underline{P}$ and note then that $Z \cap L_\beta$ is contained in $L \cap \langle N, \overline{N} \rangle$ for all $\beta \in \Delta$ that do not occur in L .) In that case $(\mathfrak{n}\text{-Ind}_{\underline{P}}^G \sigma \delta_{\underline{P}}^{-1/2})^{\text{sm}} = (\text{Ind}_{\underline{P}}^G \sigma)^{\text{sm}}$ is obviously reducible. So by Proposition 3.5 the converse of the proposition is true as well.

Proof. By Proposition 3.5 there exists $s \in N_G(L) \setminus L$ and $\sigma \circ \text{Ad}(s) \cong \sigma$. Let α be the unique element of $\Delta(\underline{P}, \underline{A}_L)$. Let $r_P(\pi) := \pi_N \delta_P^{-1/2}$ be the normalized Jacquet module. Then by the geometric lemma, $r_P((\mathfrak{n}\text{-Ind}_{\underline{P}}^G \sigma \delta_{\underline{P}}^{1/2})^{\text{sm}}) \cong \sigma \delta_P^{-1/2} \oplus \sigma \delta_P^{1/2}$. Since the supercuspidal support of any subquotient of $(\mathfrak{n}\text{-Ind}_{\underline{P}}^G \sigma \delta_{\underline{P}}^{1/2})^{\text{sm}}$ is L , any

subquotient has a nonzero Jacquet module. Hence $(\mathfrak{n}\text{-Ind}_P^G \sigma \delta_P^{1/2})^{\text{sm}}$ has length two and the normalized Jacquet modules of the irreducible subquotients are $\sigma \delta_P^{-1/2}$ and $\sigma \delta_P^{1/2}$, respectively. By Casselman's criterion of square-integrability [Cas, Theorem 4.4.6], one of them is square-integrable and let π be the other irreducible subquotient. Then $r_P(\pi)$ is isomorphic to $\sigma \delta_P^{-1/2}$. Let $\theta: \pi \rightarrow \pi_N \cong \sigma$ be the natural projection.

Let $K_0 \subset G$ be a compact open subgroup which has Iwahori decomposition $K_0 = (K_0 \cap \overline{N})(K_0 \cap L)(K_0 \cap N)$ and $\sigma^{K_0 \cap L} \neq 0$. Normalize Haar measure on K_0 such that the volume of K_0 is 1 and set $e_{K_0} := \int_{K_0} \pi(k) dk$. For $c > 0$ put $A_L^{c,-} := \{a \in A_L \mid |\alpha(a)|_F \leq c\}$. Then by a result of Casselman there exists $c \leq 1$ such that

- (i) for any $a \in A_L^{c,-}$ the space $e_{K_0}(\pi(a)\pi^{K_0})$ does not depend on a [Cas, Proposition 4.1.6]; we denote it by $\pi_{A_L^{c,-}}^{K_0}$;
- (ii) the map θ gives an isomorphism $\pi_{A_L^{c,-}}^{K_0} \xrightarrow{\sim} \sigma^{K_0 \cap L}$ [Cas, Proposition 4.1.4];
- (iii) for $a \in A_L^{c,-}$, $e_{K_0} \circ \pi(a)$ preserves $\pi_{A_L^{c,-}}^{K_0}$ and for $v \in \pi_{A_L^{c,-}}^{K_0}$, we have $\theta(e_{K_0} \circ \pi(a)v) = \sigma(a)\theta(v)$ [Cas, Lemma 4.1.1].

Let ω be the central character of σ . Then ω is unitary and the conclusion in (iii) can also be written as $\theta(e_{K_0} \circ \pi(a)v) = \omega(a)\theta(v)$ for any $a \in A_L^{c,-}$.

By a result of Tadić [Tad88], π is unitary. Fix a nonzero G -invariant inner product (\cdot, \cdot) on π . Let $v \in \pi_{A_L^{c,-}}^{K_0}$ be a nonzero element and $a \in A_L^{c,-}$. Then by $(e_{K_0} \circ \pi(a)v, v) = \omega(a)(v, v)$, we have

$$\int_{K_0} (\pi(ka)v, v)\omega(a)^{-1}(v, v)^{-1} dk = 1.$$

By the Cauchy–Schwarz inequality and since π, ω are unitary, we have

$$|(\pi(ka)v, v)\omega(a)^{-1}(v, v)^{-1}| \leq 1.$$

Hence $(\pi(ka)v, v)\omega(a)^{-1}(v, v)^{-1} = 1$ for any $k \in K_0$. In particular $(\pi(a)v, v) = \omega(a)(v, v)$. Again by the Cauchy–Schwarz inequality, we get $\pi(a)v = \omega(a)v$ for any $a \in A_L^{c,-}$. The subset $A_L^{c,-}$ generates A_L as a group. Hence $\pi(a)v = \omega(a)v$ for any $a \in A_L$.

Let $n \in N$. Then there exists $a \in A_L$ such that ana^{-1} fixes v . As $\pi(a)v = \omega(a)v$, we have $\pi(n)v = v$. By the same argument $\pi(\bar{n})v = v$ for any $\bar{n} \in \overline{N}$. Therefore $\pi^{\langle N, \overline{N} \rangle} \neq 0$. Since $\langle N, \overline{N} \rangle$ is normalized by L , it is also normalized by $G = \langle N, L, \overline{N} \rangle$. Hence π is trivial on $\langle N, \overline{N} \rangle$ by irreducibility of π . Since $\sigma \cong \pi_N$, σ is trivial on $L \cap \langle N, \overline{N} \rangle$. \square

Finally, the following criterion will be useful. We say that σ is G -regular if for $g \in N_G(L) \setminus L$, we have $\sigma \circ \text{Ad}(g) \not\cong \sigma$.

Proposition 3.8 ([Sil79, Theorem 5.4.3.7]). *If σ is supercuspidal and G -regular, then $(\mathfrak{n}\text{-Ind}_P^G \sigma)^{\text{sm}}$ is reducible if and only if μ^G has a pole at σ .*

3.2. Split groups. We now prove Theorems 1.2 and 1.3.

Theorem 3.9. *Let $G = \text{GL}_n(F)$, B the upper-triangular Borel subgroup, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z = (F^\times)^n \rightarrow C^\times$ be a continuous (hence locally \mathbb{Q}_p -analytic) character. We have $d\chi \in \text{Hom}_{\mathbb{Q}_p}(\mathfrak{z}, C) \cong \bigoplus_{\kappa: F \rightarrow C} \text{Hom}_C(\mathfrak{z} \otimes_{F, \kappa} C, C)$ and let $\lambda_\kappa = (\lambda_{\kappa,1}, \dots, \lambda_{\kappa,n})$ be the κ -component of $d\chi$, where $\lambda_{\kappa,k} \in \text{Hom}_C(C, C) \cong C$. Assume that there exists no $1 \leq i < j \leq n$ such that*

- $\lambda_{\kappa,k} - \lambda_{\kappa,k+1} \in \mathbb{Z}_{\leq 0}$ for any $k = i, \dots, j-1$ and $\kappa: F \rightarrow C$;

- $\chi_i \chi_j^{-1}(t) = |t|_F^{j-i-1} \prod_{\kappa: F \rightarrow C} \kappa(t)^{\lambda_{\kappa,i} - \lambda_{\kappa,j}}$ for all $t \in F^\times$.

Then $(\text{Ind}_B^G \chi)^{\text{cts}}$ is absolutely irreducible.

Proof. We have $\underline{G} = \text{Res}_{F/\mathbb{Q}_p} \text{GL}_n$ and the subgroup \underline{S} is the diagonal split torus of rank n over \mathbb{Q}_p . Let $e_i: \underline{S} \rightarrow \mathbb{G}_m$ be the character projecting to the i -th entry. Then the set of simple roots is $\{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$. Set $I := \{e_i - e_{i+1} \mid \lambda_{\kappa,i} - \lambda_{\kappa,i+1} \in \mathbb{Z}_{\leq 0} \text{ for all } \kappa: F \rightarrow C\}$ and \underline{Q} the standard parabolic subgroup corresponding to I . Then \underline{Q} is maximal subject to $L(-d\chi) \in \mathcal{O}^{\mathfrak{q}}$. Take a locally \mathbb{Q}_p -analytic character $\sigma_0 = \sigma_{0,1} \otimes \cdots \otimes \sigma_{0,n}: Z \rightarrow C^\times$ and a smooth character $\tau: Z \rightarrow C^\times$ such that $\chi = \sigma_0 \tau$ and if $1 \leq i \leq n-1$ and $e_i - e_{i+1} \in I$ then $(\sigma_{0,i} \sigma_{0,i+1}^{-1})(t) = \prod_{\kappa: F \rightarrow C} \kappa(t)^{\lambda_{\kappa,i} - \lambda_{\kappa,i+1}}$. Then $\underline{L}(\sigma'_0) \in \mathcal{O}^{\mathfrak{q}}$ by using Lemma 2.29. Namely, $\chi = \sigma_0 \tau$ is the decomposition given in section 2.8. By our hypothesis, $\tau_i \tau_j^{-1} \neq |\cdot|_F^{j-i-1}$ whenever $i < j$ and $-(e_i - e_j)$ is a root of Q .

We now check the assumption of Proposition 2.61. By writing L_Q as a product of general linear groups, we may assume $I = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$ (i.e. $Q = G$). Then it follows from [BZ77, 4.11 Theorem] that every nonzero subrepresentation of $(\text{Ind}_B^G \tau)^{\text{sm}}$ is generic. \square

Theorem 3.10. *Assume Assumption 2.14. Let \underline{G} be split. Let $\chi: Z \rightarrow C^\times$ be a continuous (hence locally \mathbb{Q}_p -analytic) character. We have $d\chi \in \text{Hom}_{\mathbb{Q}_p}(\mathfrak{z}, C) \cong \bigoplus_{\kappa: F \rightarrow C} \text{Hom}_C(\mathfrak{z} \otimes_{F,\kappa} C, C)$ and let $d\chi_\kappa \in X^*(\underline{Z}) \otimes C$ be the κ -component of $d\chi$. Let $\underline{P} = \underline{LN}$ be the largest standard parabolic subgroup such that $\langle d\chi_\kappa, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0}$ for all positive roots α of \underline{L} and all $\kappa: F \rightarrow C$. Assume that for all $w \in N_L(Z) \setminus Z$ there exists a root α of \underline{L} such that*

$$(3.1) \quad (\chi \delta_B^{-1/2} \circ w^{-1} \alpha^\vee) \cdot (\chi \delta_B^{-1/2} \circ \alpha^\vee)^{-1} \text{ is non-algebraic}$$

and that there exists no positive root α of \underline{L} such that

$$(3.2) \quad \chi \delta_B^{-1/2} \circ \alpha^\vee = |\cdot|_F^{-1} \prod_{\kappa: F \rightarrow C} \kappa(\cdot)^{\langle d\chi_\kappa, \alpha^\vee \rangle}.$$

Then $(\text{Ind}_B^G \chi)^{\text{cts}}$ is absolutely irreducible.

Proof. Assume first that $\underline{G}^{\text{der}}$ is simply connected. We will work over \mathbb{Q}_p by letting $\tilde{\underline{G}} := \text{Res}_{F/\mathbb{Q}_p} \underline{G}$, and likewise for $\tilde{\underline{B}}$, etc. Then $\tilde{\underline{P}}$ is the maximal standard parabolic subgroup of $\tilde{\underline{G}}$ such that $L(-d\chi) \in \mathcal{O}^{\tilde{\mathfrak{p}}}$. By Lemma 2.51 we write $\chi = \sigma_0 \tau$ with $\underline{L}(\sigma'_0) \in \mathcal{O}^{\tilde{\mathfrak{P}}}$ and τ smooth. By Lemma 2.29 we have $\sigma_0 = \sigma_{\text{alg}}(\psi|_Z)$ with $\sigma_{\text{alg}}: Z \rightarrow C^\times$ algebraic and $\psi: L \rightarrow C^\times$ locally analytic.

Fix α a positive root (of $\underline{Z} = \underline{S}$) in \underline{L} . We claim that equation (3.2) is equivalent to

$$(3.3) \quad \tau \delta_{B \cap L}^{-1/2} \circ \alpha^\vee = |\cdot|_F^{-1}.$$

Note that $\psi \circ \alpha^\vee = 1$ by Lemma 2.4. In particular, $\sigma_{\text{alg}} \circ \alpha^\vee$ is an algebraic character of F^\times with derivative $d\sigma_{\text{alg}} \circ \alpha^\vee = d\chi \circ \alpha^\vee$, so $\sigma_{\text{alg}} \circ \alpha^\vee = \prod_{\kappa: F \rightarrow C} \kappa(\cdot)^{\langle d\chi_\kappa, \alpha^\vee \rangle}$. Finally note that $\delta_B = \delta_{B \cap L} \delta_N$ and $\delta_N \circ \alpha^\vee = 1$. We deduce the claim. Likewise fix $w \in N_L(Z)$. Then $w\psi = \psi$ and $w\delta_N = \delta_N$ for all $w \in N_L(Z)$, so for $w \notin Z$ equation (3.1) becomes

$$(3.4) \quad \tau \delta_{B \cap L}^{-1/2} \circ w^{-1} \alpha^\vee \neq \tau \delta_{B \cap L}^{-1/2} \circ \alpha^\vee$$

for some root α of \underline{L} , which implies that $w(\tau \delta_{B \cap L}^{-1/2}) \neq \tau \delta_{B \cap L}^{-1/2}$.

Using equations (3.3), (3.4) we deduce that $(\text{Ind}_{B \cap L}^L \tau)^{\text{sm}} \cong (\text{n-Ind}_{B \cap L}^L \tau \delta_{B \cap L}^{-1/2})^{\text{sm}}$ has an irreducible socle that is generic by [Rod81, Proposition 1, Proposition 4]. (Alternatively we could argue using Propositions 3.22, 3.5, and 3.8, using the known reducibility points for $\text{SL}_2(F)$.) We conclude by Proposition 2.61.

For general \underline{G} , let $\varphi: \underline{G}^{\text{sc}} \rightarrow \underline{G}$ be the simply-connected cover of the derived subgroup. Let $\underline{Z}^{\text{sc}} := \varphi^{-1}(\underline{Z})$ and $\chi^{\text{sc}} := \chi \circ \varphi$. As φ is compatible with coroots of $\underline{G}^{\text{sc}}$ and \underline{G} we deduce that $d\chi^{\text{sc}}$ determines the parabolic subgroup $\varphi^{-1}(\underline{P})$ of $\underline{G}^{\text{sc}}$ by the recipe in the statement of the theorem. Moreover, conditions (3.1) and (3.2) are the same for $\underline{G}^{\text{sc}}$ and \underline{G} . We conclude by Proposition 2.42 (see Remark 2.43) and by what we already established for $\underline{G}^{\text{sc}}$. \square

Remark 3.11. This theorem generalizes to all quasisplit groups, by using Propositions 3.22, 3.5, 3.8, and the known reducibility points for $\text{SL}_2(F)$ and $\text{SU}_3(F)$. However, it can no longer be stated in terms of relative coroots. For general \underline{G} one can at least formulate a weaker version in the same way, using Remark 3.23 instead of Proposition 3.22, as well as Remark 3.61 for the possible location of reducibility points.

3.3. Classical quasisplit groups. We will now give some irreducibility theorems for all quasisplit classical groups. For simplicity we will state them for a smooth inducing character only, which is enough in light of Theorem 2.56 (at least for all but the orthogonal groups). We use standard conventions for classical groups, as for example in [Gol94], [Gol95], [CG16].

Theorem 3.12. *Let $G = \text{Sp}_{2n}(F)$ (split), B the upper-triangular Borel, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z \rightarrow C^\times$ be a smooth character. Assume $p > 2$ and the following:*

- $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{-1}$ for all $i < j$;
- $\chi_i \neq |\cdot|_F^{-1}$ for all i ;
- the set $\{\chi_i : \chi_i^2 \text{ has order two}\}$ is linearly independent over $\mathbb{Z}/2\mathbb{Z}$.

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

Note that the last condition is equivalent to saying that there do not exist (at least two) *distinct* χ_i of order 2 whose product is trivial.

Theorem 3.13. *Let $G = \text{SO}_{2n+1}(F)$ (split), B the upper-triangular Borel, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z \rightarrow C^\times$ be a smooth character. Assume $p > 2$ and the following:*

- $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{-1}$ for all $i \leq j$.

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

Theorem 3.14. *Let $G = \text{SO}_{2n}(F)$ (split), B the upper-triangular Borel, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z \rightarrow C^\times$ be a smooth character. Assume the following:*

- $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{-1}$ for all $i < j$;
- there does not exist a subset of $\{\chi_i : \chi_i^2 \text{ has order two}\}$ of even size whose product is 1.

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

Note that the last condition allows many χ_i^2 to equal 1. (Similarly below.)

Theorem 3.15. *Let $G = \text{SO}_{2n}^*(F)$ (non-split quasisplit) splitting over a quadratic extension E/F , B the upper-triangular Borel, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z = (F^\times)^{n-1} \times (E^\times)^{N_{E/F}=1} \rightarrow C^\times$ be a smooth character. Assume the following:*

- $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{-1}$ for all $i < j < n$;
- $(\chi_i \circ N_{E/F}) \chi_n^{c-1} \neq |\cdot|_E^{-1}$ for all $i < n$ if $\chi_n^2 = 1$;
- there does not exist a subset Σ of $\{\chi_i : \chi_i^2 = 1, (\chi_i \circ N_{E/F}) \chi_n^{c-1} \neq 1\}$ such that $\prod_{\chi_i \in \Sigma} \chi_i \circ N_{E/F} = \chi_n^{1-c}$ (resp. 1) if $|\Sigma|$ is odd (resp. $|\Sigma|$ is even).

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

Here, $\chi^{c-1}(x) := \chi(\bar{x}x^{-1})$ for $x \in E^\times$.

Theorem 3.16. *Let $G = \text{U}_{2n}(F)$ (quasisplit) splitting over a quadratic extension E/F , B the upper-triangular Borel subgroup, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n : Z = (E^\times)^n \rightarrow C^\times$ be a smooth character. Assume the following:*

- $\chi_i \chi_j^{-1} \neq |\cdot|_E^{-1}$ for all $i < j$;
- $\chi_i \chi_j^c \neq |\cdot|_E^{-1}$ for all $i < j$;
- $\chi_i \neq \eta |\cdot|_E^{-1/2}$ with $\eta|_{F^\times} = 1$ for all i ;
- the set $\{\chi_i : \chi_i|_{F^\times} = \omega_{E/F}\}$ has at most one element.

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

Here, $\chi^c(x) := \chi(\bar{x})$ for $x \in E^\times$, and $\omega_{E/F}$ denotes the non-trivial character of $F^\times/N_{E/F}(E^\times)$.

Theorem 3.17. *Let $G = \text{U}_{2n+1}(F)$ (quasisplit) splitting over a quadratic extension E/F , B the upper-triangular Borel subgroup, and Z the diagonal maximal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_{n+1} : Z = (E^\times)^n \times (E^\times)^{N_{E/F}=1} \rightarrow C^\times$ be a smooth character. Assume the following:*

- $\chi_i \chi_j^{-1} \neq |\cdot|_E^{-1}$ for all $i < j \leq n$;
- $\chi_i \chi_j^c \neq |\cdot|_E^{-1}$ for all $i < j \leq n$;
- $\chi_i \chi_{n+1}^{c-1} \neq |\cdot|_E^{-1}$ for all $i \leq n$;
- $\chi_i \chi_{n+1}^{c-1} \neq \eta |\cdot|_E^{-1/2}$ with $\eta|_{F^\times} = \omega_{E/F}$ for all $i \leq n$;
- $\chi_i \chi_{n+1}^{c-1} = 1$ or $\chi_i \chi_{n+1}^{c-1}|_{F^\times} \neq 1$ for all $i \leq n$.

Then $(\text{Ind}_B^G \chi \delta_B^{1/2})^{\text{cts}}$ is absolutely irreducible.

To prepare for the proofs, we first prove some lemmas about even orthogonal groups.

Lemma 3.18. *Suppose $G = \text{SO}_4^*(F)$ with diagonal maximal torus $Z = \text{GL}_1(F) \times \text{SO}_2^*(F) \cong F^\times \times (E^\times)^{N_{E/F}=1}$ and $\chi = \chi_1 \otimes \chi_2 : Z \rightarrow C^\times$. Write $\chi_1 = \psi_1 |\cdot|_F^s$ with ψ_1 unitary and $s \in \mathbb{R}$. Then the principal series $(\text{n-Ind}_B^G \chi)^{\text{sm}}$ is reducible if and only if $\psi_1^2 = \chi_2^2 = 1$ and either*

- $\psi_1 \circ N_{E/F} = \chi_2^{1-c}$ and $s = \pm 1$, or
- $\psi_1 \circ N_{E/F} \neq \chi_2^{1-c}$ and $s = 0$.

Proof. The simply-connected cover of \underline{G} is $\widetilde{G} = \text{Spin}_4^* \cong \text{Res}_{E/F} \text{SL}_2$. Note that the non-trivial Weyl group element fixes $\psi_1 \otimes \chi_2$ if and only if $\psi_1^2 = \chi_2^2 = 1$. Using a root datum calculation we verify that χ pulls back to the character $\text{diag}(x, x^{-1}) \mapsto \chi_1(x\bar{x})\chi_2(x^{-1}\bar{x})$ of the diagonal maximal torus of $\text{SL}_2(E)$. We conclude by comparison with $\text{SL}_2(E)$ using Propositions 3.6, 3.5. \square

Lemma 3.19. *Suppose $G = \text{GSO}_4^*(F)$ with diagonal maximal torus $Z = \text{GL}_1(F) \times \text{GSO}_2^*(F) \cong F^\times \times E^\times$ and $\chi = \chi_1 \otimes \chi_2 : Z \rightarrow C^\times$. Write $\chi_1 = \psi_1 |\cdot|_F^{s_1}$, $\chi_2 = \psi_2 |\cdot|_E^{s_2}$ with ψ_i unitary and $s_i \in \mathbb{R}$. Then the principal series $(\text{n-Ind}_B^G \chi)^{\text{sm}}$ is reducible if and only if $\psi_1^2 = 1$, $\psi_1 \circ N_{E/F} = \psi_2^{1-c}$, and $s_1 = \pm 1$.*

Proof. The non-trivial Weyl group element fixes $\psi_1 \otimes \psi_2$ if and only if $\psi_1^2 = 1$ and $\psi_1 \circ N_{E/F} = \psi_2^{1-c}$. In this case we determine the reducibility point by restriction to $\text{SO}_4^*(F)$ (using Propositions 3.6, 3.5, and Lemma 3.18). \square

Lemma 3.20. *Suppose $G = \text{GSO}_{2n}(F)$. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n \otimes \rho: Z \rightarrow \mathbb{C}^\times$ be a unitary smooth character. Then the R -group $R(\chi)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where 2^r is the number of subsets $\Sigma \subset \{\chi_i : \chi_i^2 = 1\}$ such that $|\Sigma|$ is even and $\prod_{\chi_i \in \Sigma} \chi_i = 1$.*

Proof. Keys' argument still applies as in [Gol94, Lemma 6.7] to show that any element of $R(\chi)$ is a product of (an even number of) sign changes. We may use the Weyl group to assume that there exist $r_1 < r_2 < \cdots < r_k$ such that $\chi_i^2 = 1$ if and only if $i \leq r_k$, $\chi_i = \chi_{i+1}$ for all $i < r_k$ such that $i \notin \{r_1, \dots, r_k\}$, and χ_{r_j} ($1 \leq j \leq k$) are pairwise distinct. Let c_i be the element in the Weyl group which changes the i -th sign. Then it is straightforward to show that $R(\chi)$ consists of all elements $\prod_{i \in \Sigma'} c_i$, where $\Sigma' \subset \{r_1, \dots, r_k\}$, $|\Sigma'|$ is even, and $\prod_{i \in \Sigma'} \chi_i = 1$. This implies the result. \square

Lemma 3.21. *Suppose $G = \text{GSO}_{2n}^*(F)$. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z = (F^\times)^{n-1} \times E^\times \rightarrow \mathbb{C}^\times$ be a unitary smooth character. Then the R -group $R(\chi)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where 2^r is the number of subsets Σ of $\{\chi_i : \chi_i^2 = 1, (\chi_i \circ N_{E/F})\chi_n^{c-1} \neq 1\}$ such that $\prod_{\chi_i \in \Sigma} \chi_i \circ N_{E/F} = \chi_n^{1-c}$ (resp. $\prod_{\chi_i \in \Sigma} \chi_i \circ N_{E/F} = 1$) if $|\Sigma|$ is odd (resp. $|\Sigma|$ is even).*

Proof. By Lemma 3.19 we see that the set Δ' in the definition of the R -group [Gol94, §1] consists of all positive roots $e_i - e_j$ if $\chi_i = \chi_j$ ($i < j < n$), $e_i + e_j$ if $\chi_i = \chi_j^{-1}$ ($i < j < n$), e_i if $\chi_i^2 = 1$ and $(\chi_i \circ N_{E/F})\chi_n^{c-1} = 1$ ($i < n$). Keys' argument still applies as in [CG16, Lemma A.2] to show that any element of $R(\chi)$ is a product of sign changes. We may use the Weyl group to assume that there exist $r_1 < r_2 < \cdots < r_k$ such that $\chi_i^2 = 1$ and $(\chi_i \circ N_{E/F})\chi_n^{c-1} \neq 1$ if and only if $i \leq r_k$, $\chi_i = \chi_{i+1}$ for all $i < r_k$ such that $i \notin \{r_1, \dots, r_k\}$, and χ_{r_j} ($1 \leq j \leq k$) are pairwise distinct. Then it is straightforward to show that $R(\chi)$ consists of all elements $\prod_{i \in \Sigma'} c_i$, where $\Sigma' \subset \{r_1, \dots, r_k\}$, and $\prod_{i \in \Sigma'} \chi_i \circ N_{E/F} = 1$ if $|\Sigma'|$ is even (resp. $\prod_{i \in \Sigma'} \chi_i \circ N_{E/F} = \chi_n^{1-c}$ if $|\Sigma'|$ is odd). This implies the result. \square

Proposition 3.22. *Suppose \underline{G} is quasisplit and $\chi: Z \rightarrow \mathbb{C}^\times$ a smooth character. Write $\chi = \psi_{\chi_\nu}$ with ψ unitary and $\nu \in \mathfrak{a}_{Z, \mathbb{R}}^*$. Let L be the maximal semistandard Levi subgroup such that $\nu \in \mathfrak{a}_{L, \mathbb{R}}^*$. Then the following two conditions are equivalent:*

- (i) *the socle of $(\text{n-Ind}_B^G \chi)^{\text{sm}}$ is of the same length as (the semisimple representation) $(\text{n-Ind}_{B \cap L}^L \chi)^{\text{sm}}$, and every irreducible subrepresentation of $(\text{n-Ind}_B^G \chi)^{\text{sm}}$ is generic;*
- (ii) (a) *for all $\alpha \in \Phi_{\text{red}}^+$ such that $(\text{n-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\text{sm}}$ is reducible we have $\langle \nu, \alpha^\vee \rangle \geq 0$;*
 (b) *every irreducible subrepresentation of $(\text{n-Ind}_{B \cap L}^L \chi)^{\text{sm}}$ is generic.*

Proof. (We thank Alberto Mínguez for providing the key ideas for this argument.) First assume that (i) holds. Note first that by [Rod73, Theorem 2] in the split case and by the geometric lemma [BZ77, 5.2 Theorem] in general, for each non-degenerate character θ of U there is a unique θ -generic irreducible constituent of $(\text{n-Ind}_B^G \chi)^{\text{sm}}$. Suppose first that there is a parabolic subgroup $\underline{P}' = \underline{L}' \underline{N}'$ such that $0 \rightarrow \tau_1 \rightarrow (\text{n-Ind}_{B \cap L'}^{L'} \chi)^{\text{sm}} \rightarrow \tau_2 \rightarrow 0$ with $\tau_1 \neq 0$ non-generic. Then every irreducible submodule of $(\text{n-Ind}_{\underline{P}'}^G, \tau_1)^{\text{sm}}$ is non-generic (as the generic constituents have to lie in $(\text{n-Ind}_{\underline{P}'}^G, \tau_2)^{\text{sm}}$), contradiction. Therefore, every irreducible subrepresentation of $(\text{n-Ind}_{B \cap L'}^{L'} \chi)^{\text{sm}}$ is generic. In particular this applies to $L' = L$. Consider now all $L' = L_\alpha$ for $\alpha \in \Phi_{\text{red}}^+$ such that $(\text{n-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\text{sm}}$ is reducible and $\langle \nu, \alpha^\vee \rangle < 0$. Then $(\text{n-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\text{sm}}$ is of length 2 and has an irreducible socle (by considering the Jacquet module). We claim that the socle is non-generic. We get

the same length 2 reducibility after restriction to the simply-connected cover of the derived subgroup of L_α , which is one of $\mathrm{SL}_2(E)$, $\mathrm{SU}_3(E)$ for some finite extension E/F . In case of $\mathrm{SL}_2(E)$ we are done because the subrepresentation is trivial by the condition $\langle \nu, \alpha^\vee \rangle < 0$. In the other case we lift to $\mathrm{U}_3(E)$ and still obtain reducibility (e.g. by Proposition 3.6). Relabeling E as F , we reduce to $G = \mathrm{U}_3(F)$ and $\langle \nu, \alpha^\vee \rangle < 0$ for the unique simple root α . Then the socle is non-generic, e.g. by (the contragredient of) [CS98, Theorem 1]. (Note that there is only one orbit of non-degenerate characters of U_3 .) Thus we deduce (ii).

Conversely, suppose that (ii) holds. We first show that the socle of $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}}$ is of the same length as $(\mathrm{n}\text{-Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$. If ν is not dominant, then there is an $\alpha \in \Delta$ such that $\langle \nu, \alpha^\vee \rangle < 0$. By the first assumption, $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$ is irreducible, so we deduce that $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_B^G s_\alpha(\chi))^{\mathrm{sm}}$ by transitivity of parabolic induction. Since s_α permutes the set $\Phi_{\mathrm{red}}^+ \setminus \{\alpha\}$, our condition on χ above also holds for $s_\alpha(\chi) = s_\alpha(\psi)\chi_{s_\alpha(\nu)}$. But in this way we reduce the number of $\alpha \in \Phi_{\mathrm{red}}^+$ such that $\langle \nu, \alpha^\vee \rangle < 0$ by one, and we can reduce to ν dominant in finitely many steps.

Suppose now that ν is dominant, as we may. Then \underline{L} is standard and the standard parabolic subgroup \underline{P} containing \underline{L} as a Levi part is associated to the subset $\{\alpha \in \Delta : \langle \nu, \alpha^\vee \rangle = 0\}$ of Δ . As ν lies in $\mathfrak{a}_{L, \mathbb{R}}^*$, χ_ν extends to an unramified character of L . Let σ be any irreducible constituent of $(\mathrm{n}\text{-Ind}_{B \cap L}^L \chi)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{B \cap L}^L \psi)^{\mathrm{sm}} \otimes \chi_\nu$, which is a semisimple representation (as it is unitary up to twist). Then σ is generic by our second assumption, and we now show that $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ has an irreducible socle. By the geometric lemma, $r_P((\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}})$ has a filtration with graded pieces $\sigma_w := (\mathrm{n}\text{-Ind}_{L \cap w P w^{-1}}^L w r_{L \cap w^{-1} P w} \sigma)^{\mathrm{sm}}$, where r_P denotes the normalized Jacquet module and w runs through Kostant representatives of $W_L \backslash W / W_L$. Note that the supercuspidal support of σ (resp. of any irreducible constituent of σ_w) is (Z, χ) (resp. $(Z, w w' \chi)$ for some $w' \in W_L$), up to L -conjugacy. But $w \chi = \chi$ implies that $w \chi_\nu = \chi_\nu$, hence $w \nu = \nu$, i.e. $w \in W_L$ by definition of L . We see that σ occurs with multiplicity one in $r_P((\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}})$ and hence by exactness of r_P deduce that $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ has an irreducible socle. (If $\pi \subset (\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ is irreducible, then $r_P \pi \twoheadrightarrow \sigma$.)

It remains to show that every irreducible subrepresentation of $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}}$ is generic. Continue to assume, as we may, that ν is dominant. Taking σ to be any irreducible constituent of the semisimple representation $(\mathrm{n}\text{-Ind}_{B \cap L}^L \chi)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{B \cap L}^L \psi)^{\mathrm{sm}} \otimes \chi_\nu$ it suffices to show that the irreducible socle π_σ of $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ is generic. Fix a non-degenerate character θ such that $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ contains a θ -generic constituent π_θ . We claim that there exists a (nonzero) intertwining morphism $T: (\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ whose kernel does not contain π_θ as constituent. Assuming the claim we are done: by second adjointness and the previous paragraph we know that $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ has irreducible cosocle π_σ . (If $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}} \twoheadrightarrow \pi'$ irreducible, then $\sigma \hookrightarrow r_P \pi'$, so π' has to be the unique constituent with σ contributing to $r_P \pi'$. In fact, we see that σ is a direct summand of $r_P \pi_\sigma$.) Then the image of T has π_σ both as its socle and its cosocle, so the image of T equals π_σ , since π_σ occurs only once in $(\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$. The claim implies that $\pi_\sigma \cong \pi_\theta$ is generic.

To prove the final claim, it suffices to construct a morphism $T': (\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}}$ that does not kill any generic constituents, because we can decompose $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}} \cong \bigoplus_\sigma (\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ and $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}} \cong \bigoplus_\sigma (\mathrm{n}\text{-Ind}_P^G \sigma)^{\mathrm{sm}}$ (by semisimplicity) and consider where π_σ is sent. By writing w_0 as a reduced product of simple reflections, it suffices to construct $(\mathrm{n}\text{-Ind}_{w^{-1} s_\beta^{-1} B s_\beta w}^G \chi)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_{w^{-1} B w}^G \chi)^{\mathrm{sm}}$

that does not kill any generic constituents, for any $w \in W$ and simple reflection s_β with $\ell(s_\beta w) > \ell(w)$ (i.e. $w^{-1}(\beta) > 0$) where ℓ is the length function on W . By parabolic induction, it suffices to construct $(\mathrm{n}\text{-Ind}_{w^{-1}s_\beta^{-1}B s_\beta w}^{w^{-1}L_\beta w} \chi)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_{w^{-1}B w}^{w^{-1}L_\beta w} \chi)^{\mathrm{sm}}$ that does not kill any generic constituents. We can write this morphism as $T'_\alpha: (\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$, where $\alpha := w^{-1}(\beta) \in \Phi_{\mathrm{red}}^+$. If $\langle \nu, \alpha^\vee \rangle = 0$ or if $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$ is irreducible, these representations are semisimple and we take T'_α to be any isomorphism. Otherwise, by (a) we know that $\langle \nu, \alpha^\vee \rangle > 0$. We also know that $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$ has an irreducible cosocle which is the same as the socle of $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$, and we take T'_α to be the unique (up to scalar) nonzero map possible. By the condition that $\langle \nu, \alpha^\vee \rangle > 0$ we know that the kernel of T'_α is the unique non-generic constituent of $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \chi)^{\mathrm{sm}}$ (see the first paragraph of this proof). This completes the proof. \square

Remark 3.23. We have a similar criterion for irreducibility, for any connected reductive group \underline{G} . Suppose that σ is a (finite-dimensional) irreducible smooth representation of the minimal Levi subgroup \underline{L} . Write $\sigma \cong \sigma_u \chi_\nu$ with σ_u unitary and $\nu \in \mathfrak{a}_{\mathbb{Z}, \mathbb{R}}^*$, and let \underline{L} be the maximal semistandard Levi subgroup such that $\nu \in \mathfrak{a}_{\underline{L}, \mathbb{R}}^*$. Then $(\mathrm{n}\text{-Ind}_{\underline{B}}^{\underline{G}} \sigma)^{\mathrm{sm}}$ is irreducible if and only if $(\mathrm{n}\text{-Ind}_{\underline{B} \cap \underline{L}_\alpha}^{\underline{L}_\alpha} \sigma)^{\mathrm{sm}}$ is irreducible for all $\alpha \in \Phi_{\mathrm{red}}^+$ and $(\mathrm{n}\text{-Ind}_{\underline{B} \cap \underline{L}}^{\underline{L}} \sigma)^{\mathrm{sm}}$ is irreducible. For the “if” direction, the first condition shows that $(\mathrm{n}\text{-Ind}_{\underline{B}}^{\underline{G}} \sigma)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{\underline{B}}^{\underline{G}} w(\sigma))^{\mathrm{sm}}$ for any $w \in W$ (by induction on $\ell(w)$). In particular, $(\mathrm{n}\text{-Ind}_{\underline{B}}^{\underline{G}} \sigma)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{\underline{B}}^{\underline{G}} \sigma)^{\mathrm{sm}}$ and we may also assume that ν is dominant. Let $\tau := (\mathrm{n}\text{-Ind}_{\underline{B} \cap \underline{L}}^{\underline{L}} \sigma)^{\mathrm{sm}}$. Then $(\mathrm{n}\text{-Ind}_{\underline{P}}^{\underline{G}} \tau)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{\underline{P}}^{\underline{G}} \tau)^{\mathrm{sm}}$, where $\underline{P} = \underline{L}\underline{N}$ is the standard parabolic with Levi subgroup \underline{L} . By the same argument as in the proof of Proposition 3.22 we see that $(\mathrm{n}\text{-Ind}_{\underline{P}}^{\underline{G}} \tau)^{\mathrm{sm}}$ has an irreducible socle which is also the cosocle of $(\mathrm{n}\text{-Ind}_{\underline{P}}^{\underline{G}} \tau)^{\mathrm{sm}}$ and occurs in these representations with multiplicity one. This implies irreducibility. (If \underline{G} is split, this can also be deduced from [Mul79, Proposition 4.2], and in general from the main theorem of [Luo19].)

Proof of Theorems 3.12–3.17. By Proposition 2.61 it suffices to verify that both assumptions in Proposition 3.22(ii) hold. (To apply this proposition we work with \underline{G} obtained by restriction of scalars from F to \mathbb{Q}_p , and take $\sigma_0 = 1$, $\tau = \chi \delta_B^{1/2}$, $\underline{Q} = \underline{G}$. We also embed C in \mathbb{C} and replace C by a finite extension so that all irreducible constituents of $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ are absolutely irreducible.) Note that $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}} = (\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}}$. The first assumption is easy to verify by the irreducibility criteria for smooth principal series of $\mathrm{Sp}_2(F) = \mathrm{SL}_2(F)$, $\mathrm{SO}_3(F) \cong \mathrm{PGL}_2(F)$, $\mathrm{GL}_2(E)$, $\mathrm{SO}_4^*(F)$, $\mathrm{U}_2(F)$ and $\mathrm{U}_3(F)$, using all but the last hypothesis in each theorem. (In this proof we consider Theorem 3.13 to have a vacuous last hypothesis. For the last three irreducibility criteria, see Lemma 3.18, respectively [Rog90, §11.1, §12.2].)

We now verify the second assumption of Proposition 3.22, using the last hypothesis in each theorem. Write $L \cong \prod_i \mathrm{GL}_{m_i}(E) \times G'$, where $E = F$ in all but the unitary cases and G' is a classical group of the same type as G . (Note that G' may be trivial, except when the type is SO^* .) As unitary principal series of $\mathrm{GL}_m(E)$ are irreducible and generic, we may reduce to the case where $G' = G$, i.e. $L = G$. By a twist we may assume that χ is unitary, i.e. $\nu = 0$. In case $G = \mathrm{SO}_{2n+1}(F)$ (resp. $G = \mathrm{U}_{2n+1}(F)$) we apply [Gol94, Theorem 6.5(1)] (resp. [Gol95, Theorem 3.4]) to see that $(\mathrm{n}\text{-Ind}_B^G \chi)^{\mathrm{sm}}$ is irreducible and hence generic.

In case $G = \mathrm{Sp}_{2n}(F)$ we lift χ to the character $\tilde{\chi} := \chi_1 \otimes \cdots \otimes \chi_n \otimes \rho$ of the diagonal maximal torus \tilde{Z} of $\mathrm{GSp}_{2n}(F)$, where $\rho: F^\times \rightarrow \mathbb{C}^\times$ is an arbitrary unitary character. The unitary principal series of $\mathrm{GSp}_{2n}(F)$ obtained from $\tilde{\chi}$ is irreducible

and hence generic by [Gol97, Theorem 2.6]. (Note that $d_1 = 0$ and $d_\chi \leq 1$ for all $\chi \neq 1$ in the notation of that paper.) By restriction to $\mathrm{Sp}_{2n}(F)$ it follows that all irreducible constituents of $(\mathrm{n}\text{-}\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ are generic.

In case $G = \mathrm{U}_{2n}(F)$ we lift χ to the character $\tilde{\chi} := \chi_1 \otimes \cdots \otimes \chi_n \otimes \rho$ of the diagonal maximal torus \tilde{Z} of $\mathrm{GU}_{2n}(F)$, where $\rho : F^\times \rightarrow \mathbb{C}^\times$ is an arbitrary unitary character. The unitary principal series of $\mathrm{GSp}_{2n}(F)$ obtained from $\tilde{\chi}$ is irreducible and hence generic by [Gol97, Theorem 2.6]. (Note that $d_1 = 0$ and $\Lambda(\sigma) = \Lambda(\sigma)' = \emptyset$ in the notation of that paper. Here we use that by comparison with $\mathrm{SL}_2(F)$, using Proposition 3.6, all unitary principal series of $\mathrm{GU}_2(F)$ are irreducible.) By restriction to $\mathrm{U}_{2n}(F)$ it follows that all irreducible constituents of $(\mathrm{n}\text{-}\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ are generic.

In case $G = \mathrm{SO}_{2n}(F)$ (resp. $G = \mathrm{SO}_{2n}^*(F)$) we likewise lift to a character of the diagonal maximal torus \tilde{Z} of $\mathrm{GSO}_{2n}(F)$ and apply Lemma 3.20 (resp. Lemma 3.21). \square

Remark 3.24. The conditions in Theorems 3.12–3.17 are optimal in the sense that if every irreducible subrepresentation of $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is generic, then the conditions in the theorems hold. (The analogue of course holds for $\mathrm{GL}_n(F)$ as well, by the same argument.)

To justify this, suppose that every irreducible subrepresentation of $(\mathrm{n}\text{-}\mathrm{Ind}_B^G \chi)^{\mathrm{sm}}$ is generic. Then by Proposition 3.22 (item (a)) we get all but the last condition in each theorem.

It remains to discuss the symplectic, even orthogonal, and unitary groups. Observe that there is a unique orbit of non-degenerate characters of U for the groups $\mathrm{GSp}_{2n}(F)$, $\mathrm{GSO}_{2n}(F)$, $\mathrm{GSO}_{2n}^*(F)$, $\mathrm{GU}_{2n}(F)$, and $\mathrm{U}_{2n+1}(F)$ under the action of the diagonal maximal torus \underline{Z} . If $G = \mathrm{U}_{2n+1}(F)$ we let \underline{L} be the maximal semi-standard Levi subgroup such that $\nu \in \mathfrak{a}_{L, \mathbb{R}}^*$. Then the semisimple representation $(\mathrm{n}\text{-}\mathrm{Ind}_{B \cap L}^L \chi)^{\mathrm{sm}}$ has to be irreducible by (b) (as it contains a unique generic constituent), which implies the last condition in this case by the R -group result we already used. If G is one the groups $\mathrm{Sp}_{2n}(F)$, $\mathrm{SO}_{2n}(F)$, $\mathrm{SO}_{2n}^*(F)$, $\mathrm{U}_{2n}(F)$ we lift to the similitude group $\mathrm{GSp}_{2n}(F)$, $\mathrm{GSO}_{2n}(F)$, $\mathrm{GSO}_{2n}^*(F)$, $\mathrm{GU}_{2n}(F)$ as in the proof above and then apply the same reasoning. (Note that the lifted principal series still has the property that every irreducible subrepresentation is generic.)

For completeness, we also state the irreducibility criteria we get from Remark 3.23 for classical groups, using the same notation as in Theorems 3.12–3.17. For the group $\mathrm{Sp}_{2n}(F)$, see also [Tad94, Theorem 7.1].

Theorem 3.25.

- (i) If $G = \mathrm{Sp}_{2n}(F)$ (split), $\chi = \chi_1 \otimes \cdots \otimes \chi_n : Z \rightarrow \mathbb{C}^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
 - $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{\pm 1}$ for all $i < j$;
 - $\chi_i \neq |\cdot|_F^{\pm 1}$ for all i ;
 - χ_i is not of order 2 for all i .
- (ii) If $G = \mathrm{SO}_{2n+1}(F)$ (split), $\chi = \chi_1 \otimes \cdots \otimes \chi_n : Z \rightarrow \mathbb{C}^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
 - $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{\pm 1}$ for all $i \leq j$.
- (iii) If $G = \mathrm{SO}_{2n}(F)$ (split), $\chi = \chi_1 \otimes \cdots \otimes \chi_n : Z \rightarrow \mathbb{C}^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
 - $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{\pm 1}$ for all $i < j$;
 - the set $\{\chi_i : \chi_i^2 = 1\}$ has at most one element.

- (iv) If $G = \mathrm{SO}_{2n}^*(F)$ (non-split quasisplit) splitting over a quadratic extension E/F , $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z = (F^\times)^{n-1} \times (E^\times)^{N_{E/F}=1} \rightarrow C^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
- $\chi_i \chi_j^{\pm 1} \neq |\cdot|_F^{\pm 1}$ for all $i < j < n$;
 - $(\chi_i \circ N_{E/F}) \chi_n^{c-1} \neq |\cdot|_E^{\pm 1}$ for all $i < n$ if $\chi_n^2 = 1$;
 - the set $\{\chi_i : i < n \text{ and } \chi_i^2 = 1\}$ has at most one element if $\chi_n^2 \neq 1$;
 $(\chi_i^2 \neq 1 \text{ or } (\chi_i \circ N_{E/F}) \chi_n^{c-1} = 1)$ for all $i < n$ if $\chi_n^2 = 1$.
- (v) If $G = \mathrm{U}_{2n}(F)$ (quasisplit) splitting over a quadratic extension E/F , $\chi = \chi_1 \otimes \cdots \otimes \chi_n: Z = (E^\times)^n \rightarrow C^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
- $\chi_i \chi_j^{-1} \neq |\cdot|_E^{\pm 1}$ for all $i < j$;
 - $\chi_i \chi_j^c \neq |\cdot|_E^{\pm 1}$ for all $i < j$;
 - $\chi_i \neq \eta |\cdot|_E^{\pm 1/2}$ with $\eta|_{F^\times} = 1$ for all i ;
 - $\chi_i|_{F^\times} \neq \omega_{E/F}$ for all i .
- (vi) If $G = \mathrm{U}_{2n+1}(F)$ (quasisplit) splitting over a quadratic extension E/F , $\chi = \chi_1 \otimes \cdots \otimes \chi_{n+1}: Z = (E^\times)^n \times (E^\times)^{N_{E/F}=1} \rightarrow C^\times$ smooth, then $(\mathrm{Ind}_B^G \chi \delta_B^{1/2})^{\mathrm{sm}}$ is irreducible if and only if
- $\chi_i \chi_j^{-1} \neq |\cdot|_E^{\pm 1}$ for all $i < j \leq n$;
 - $\chi_i \chi_j^c \neq |\cdot|_E^{\pm 1}$ for all $i < j \leq n$;
 - $\chi_i \chi_{n+1}^{c-1} \neq |\cdot|_E^{\pm 1}$ for all $i \leq n$;
 - $\chi_i \chi_{n+1}^{c-1} \neq \eta |\cdot|_E^{\pm 1/2}$ with $\eta|_{F^\times} = \omega_{E/F}$ for all $i \leq n$;
 - $\chi_i \chi_{n+1}^{c-1} = 1$ or $\chi_i \chi_{n+1}^{c-1}|_{F^\times} \neq 1$ for all $i \leq n$.

Proof. This follows as in the proof of Theorems 3.12–3.17. If $G = \mathrm{Sp}_{2n}(F)$ (resp. $\mathrm{SO}_{2n}(F)$, resp. $\mathrm{SO}_{2n}^*(F)$, resp. $G = \mathrm{U}_{2n}(F)$), the relevant R -group result can be found in [Gol94, Theorem 6.4] (resp. [Gol94, Theorem 6.8], resp. [CG16, Theorem A.4], resp. [Gol95, Theorem 3.4]). \square

3.4. The group $\mathrm{GL}_n(D)$. Let D be a central division F -algebra of dimension d^2 and $G_n = \mathrm{GL}_n(D)$, $B = B_n$ the minimal parabolic subgroup of upper-triangular matrices, $U = U_n$ the unipotent radical of B_n , and $Z = Z_n$ the diagonal minimal Levi subgroup. The mirabolic subgroup P_n of $\mathrm{GL}_n(D)$ is defined by $P_n := \{(g_{ij}) \in G \mid g_{nn} = 1, g_{ni} = 0 \ (1 \leq i \leq n-1)\}$. We say that a representation π of P_n is *generic* if $\pi_{U_n, \theta} \neq 0$ for an (equivalently any) non-degenerate character θ of U_n . Let $\mathrm{Nrd}: D^\times \rightarrow F^\times$, $\det: G_n \rightarrow F^\times$ be the reduced norm. For an absolutely irreducible smooth representation σ of D^\times , let $\nu_\sigma = |\mathrm{Nrd}|_F^{s(\sigma)}$ be the character of D^\times of [Tad90, Section 2]. Here, $s(\sigma)$ is a positive integer dividing d . It is characterized by the fact that for σ' another absolutely irreducible smooth representation of D^\times the induction $(n\text{-Ind}_{B_2}^{G_2} \sigma \boxtimes \sigma')^{\mathrm{sm}}$ is reducible if and only if $\sigma' \cong \sigma \nu_\sigma^{\pm 1}$ [Tad90, 2.5 Lemma, 4.2 Lemma]. (Note that σ' does not denote the strong dual of σ in this subsection.)

Theorem 3.26. *Let $\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_n$ be an absolutely irreducible smooth representation of $Z \cong (D^\times)^n$ over C . Assume the following condition.*

(3.5) *There exists no sequence $1 \leq i_0 < i_1 < \cdots < i_e \leq n$ such that*

$$\sigma_{i_j} \cong \sigma_{i_0} \nu_{\sigma_{i_0}}^j \text{ for all } 0 \leq j \leq e \text{ and } \nu_{\sigma_{i_0}}^e = |\mathrm{Nrd}|_F^d.$$

Then any nonzero P_n -subrepresentation of $(n\text{-Ind}_B^G \sigma)^{\mathrm{sm}}$ is generic. In particular, $(\mathrm{Ind}_B^G \sigma \delta_B^{1/2})^{\mathrm{cts}}$ is irreducible.

The last part follows from Proposition 2.61 for the group $\mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_n(D)$ (and $\sigma_0 = 1$, $\tau = \sigma$, $Q = G$).

We fix an absolutely irreducible smooth Z -representation $\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_n$ satisfying (3.5). We first claim that $(\mathrm{n}\text{-Ind}_B^G \sigma)^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_B^G \tau)^{\mathrm{sm}}$ for some $\tau = \tau_1 \boxtimes \cdots \boxtimes \tau_n$ in the Weyl group orbit of σ such that $\tau_i \not\cong \tau_j \otimes |\mathrm{Nrd}|_F^{-d}$ for any $i < j$. Letting \sim be the equivalence relation on absolutely irreducible smooth representations of D^\times induced by $\zeta \sim \zeta \nu_\zeta$. As $(\mathrm{n}\text{-Ind}_{B_2}^{G_2} \sigma_i \boxtimes \sigma_{i+1})^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{B_2}^{G_2} \sigma_{i+1} \boxtimes \sigma_i)^{\mathrm{sm}}$ whenever $\sigma_i \not\sim \sigma_{i+1}$, by repeated transposition of consecutive, inequivalent representations we may assume that whenever $\sigma_i \sim \sigma_j$ for some $i < j$, then $\sigma_i \sim \sigma_{i+1} \sim \cdots \sim \sigma_j$, while preserving condition (3.5). In this way we reduce to the case where all σ_i ($1 \leq i \leq n$) lie in the same equivalence class, i.e. $\sigma_i \cong \zeta \nu_\zeta^{k(i)}$ for some $k(i) \in \mathbb{Z}$. Let e be the divisor of d such that $\nu_\zeta^e = |\mathrm{Nrd}|_F^d$. Condition (3.5) says that there exists no sequence $1 \leq i_0 < i_1 < \cdots < i_e \leq n$ such that $k(i_j) - k(i_0) = j$ for all $0 \leq j \leq e$. If there exists any $0 \leq \ell < n$ such that $k(\ell+1) - k(\ell) \geq 2$, then we can transpose σ_ℓ and $\sigma_{\ell+1}$, while keeping condition (3.5) satisfied. (Subsequences $i_0 < \cdots < i_e$ as in (3.5) containing at most one of $\ell, \ell+1$ are unaffected by the transposition. Subsequences containing both will no longer satisfy (3.5) after transposition because the function $k(i_j)$ has to be increasing.) Each such transposition decreases the sum $\sum_{\ell=1}^n \ell k(\ell)$, so after finitely many steps we may assume that $k(\ell+1) - k(\ell) \leq 1$ for all $0 \leq \ell < n$ and that condition (3.5) still holds. Suppose there exist $i < j$ such that $\sigma_i \cong \sigma_j \otimes |\mathrm{Nrd}|_F^{-d}$, i.e. $k(j) - k(i) = e$. We have $e > 1$ by condition (3.5). More generally, suppose that for some $i < j$ we have $k(j) - k(i) = e' > 1$. Pick $i \leq i' < j$ maximal such that $k(i') - k(i) \leq 1$, so $k(i'+1) - k(i) \geq 2$. Then $1 \geq k(i'+1) - k(i') \geq 2 - 1 = 1$, whence equality holds and in particular $k(i') - k(i) = 1$. Hence, taking again $e' = e$, then by induction there exists a sequence $i = i_0 < i_1 < \cdots < i_e = j$ such that $k(i_j) - k(i_{j-1}) = 1$ for all $1 \leq j \leq e$, contradiction.

Therefore, we may assume that $\sigma_i \not\cong \sigma_j \otimes |\mathrm{Nrd}|_F^{-d}$ for any $i < j$. If $D = F$ then the theorem is proved in [BZ77, 4.11 Theorem] and in general we follow their argument.

For a locally profinite topological group H , let $\mathrm{Rep}^\infty(H)$ be the category of smooth H -representations over C . If H' is a closed subgroup of H and $\tau \in \mathrm{Rep}^\infty(H)$, we put $(\mathrm{n}\text{-Ind}_{H'}^H \tau)^{\mathrm{sm}} := (\mathrm{Ind}_{H'}^H \tau \delta_H^{-1/2} \delta_{H'}^{1/2})^{\mathrm{sm}}$. Let N_n be the unipotent radical of P_n . We fix a non-degenerate character θ of U_n and let η_n be the restriction of θ to N_n . We define exact functors as follows:

$$\begin{aligned} \Phi^- : \mathrm{Rep}^\infty(P_n) &\rightarrow \mathrm{Rep}^\infty(P_{n-1}) & \pi &\mapsto |\det|_F^{-d/2} \otimes \pi_{N_n, \eta_n}, \\ \widehat{\Phi}^+ : \mathrm{Rep}^\infty(P_{n-1}) &\rightarrow \mathrm{Rep}^\infty(P_n) & \tau &\mapsto (\mathrm{n}\text{-Ind}_{P_{n-1}N_n}^{P_n} \tau)^{\mathrm{sm}}, \\ \Phi^+ : \mathrm{Rep}^\infty(P_{n-1}) &\rightarrow \mathrm{Rep}^\infty(P_n) & \tau &\mapsto (\mathrm{nc}\text{-Ind}_{P_{n-1}N_n}^{P_n} \tau)^{\mathrm{sm}}, \end{aligned}$$

where $\mathrm{nc}\text{-Ind}$ denotes normalized compact induction. (We keep the notation of [BZ77], but caution that Φ^+ should not be confused with the set of positive roots. Also note that [BZ77] uses a different definition of normalized induction.) In the definition of $\widehat{\Phi}^+$ and Φ^+ we extend π to $P_{n-1}N_n$ by letting N_n act via η_n . We remark that $\delta_{P_{n-1}N_n} = |\det|_F^{2d}$ and $\delta_{P_n} = |\det|_F^d$, so $(\mathrm{n}\text{-Ind}_{P_{n-1}N_n}^{P_n} \tau)^{\mathrm{sm}} = (\mathrm{Ind}_{P_{n-1}N_n}^{P_n} \tau |\det|_F^{d/2})^{\mathrm{sm}}$. In particular, the functors Φ^- , $\widehat{\Phi}^+$, Φ^+ coincide with the ones in [BZ77] when $D = F$.

Lemma 3.27. *The functor Φ^- is right adjoint to Φ^+ and left adjoint to $\widehat{\Phi}^+$. We have $\Phi^-(\pi) = 0$ if and only if N_n acts trivially on π .*

Proof. By Frobenius reciprocity, $\widehat{\Phi}^+$ is the right adjoint of Φ^- . For Φ^+ , the proof of [BZ77, 3.2 Proposition (b)] applies. The last claim follows from [BZ77, 3.2 Proposition (e)] when $D = F$ and the same proof applies. \square

For a smooth representation τ , let τ^\vee be the smooth dual of τ .

Lemma 3.28. *We have $\Phi^-(\pi^\vee) \cong \Phi^-(\pi)^\vee$ for $\pi \in \text{Rep}^\infty(P_n)$.*

Proof. Let $\tau \in \text{Rep}^\infty(P_{n-1})$. We have $\Phi^+(\tau)^\vee \cong \widehat{\Phi}^+(\tau^\vee)$, by our normalization of induction. Then we have

$$\begin{aligned} \text{Hom}_{P_{n-1}}(\tau, \Phi^-(\pi^\vee)) &\cong \text{Hom}_{P_n}(\Phi^+(\tau), \pi^\vee) \\ &\cong \text{Hom}_{P_n}(\pi, \Phi^+(\tau)^\vee) \\ &\cong \text{Hom}_{P_n}(\pi, \widehat{\Phi}^+(\tau^\vee)) \\ &\cong \text{Hom}_{P_{n-1}}(\Phi^-(\pi), \tau^\vee) \\ &\cong \text{Hom}_{P_{n-1}}(\tau, \Phi^-(\pi)^\vee). \end{aligned}$$

The lemma follows. \square

Let Q_n be the standard parabolic subgroup of $G_n = \text{GL}_n(D)$ corresponding to the partition $n = 1 + (n-1)$. Then $Q_{n-1}N_n = (G_1 \times P_{n-1})N'_n$, where the normal subgroup N'_n denotes the unipotent radical of Q_n . To keep notation short we write $Q_{n-1}N_n$ below, even though we think of it as $(G_1 \times P_{n-1})N'_n$.

Lemma 3.29. *Let σ_1 be a smooth representation of G_1 and σ' a smooth representation of P_{n-1} . Suppose that $\pi_0 \subset (\text{n-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1 \boxtimes \sigma')^{\text{sm}}$ is a P_n -subrepresentation, where we let N'_n act trivially on $\sigma_1 \boxtimes \sigma'$.*

- (i) *If $n \geq 3$ we have an embedding $\Phi^-(\pi_0) \hookrightarrow (\text{n-Ind}_{Q_{n-2}N_{n-1}}^{P_{n-1}} \sigma_1 \boxtimes \Phi^-(\sigma'))^{\text{sm}}$.*
- (ii) *If $(\pi_0)_{N_n, \eta_n} = 0$ and $n \geq 2$, then $\pi_0|_{G_{n-1}}$ is embedded into the following two representations:*

$$\begin{aligned} &(\text{n-Ind}_{Q_{n-1}}^{G_{n-1}} \sigma_1 \boxtimes r(\sigma'))^{\text{sm}} \otimes |\det|_F^{d/2}, \\ &(\text{n-Ind}_{Q_{n-1}}^{G_{n-1}} \sigma_1 \boxtimes r((\sigma')^\vee)^\vee)^{\text{sm}} \otimes |\det|_F^{-d/2}. \end{aligned}$$

Here $r(\sigma') := (\sigma')_{N_{n-1}} \otimes |\det|_F^{-d/2}$ is the normalized Jacquet module of σ' with respect to the parabolic subgroup corresponding to $(n-1) = (n-2) + 1$.

Proof. For simplicity we put $\Pi := (\text{n-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1 \boxtimes \sigma')^{\text{sm}}$.

For (i), we apply the geometric lemma to calculate the P_{n-1} -representation Π_{N_n, η_n} . We have

$$\begin{aligned} Q_{n-1}N_n \backslash P_n / P_{n-1}N_n &\cong Q_{n-1} \backslash G_{n-1} / P_{n-1} \\ &\cong \text{Aut}(\{2, \dots, n-1\}) \backslash S_{n-1} / \text{Aut}(\{1, \dots, n-2\}) \end{aligned}$$

and it has two elements, represented by the identity element and $(1 \ n-1)$ (transposition). The orbit corresponding to $(1 \ n-1)$ does not contribute. Hence we have

$$\Phi^-(\Pi) \cong (\text{n-Ind}_{Q_{n-2}N_{n-1}}^{P_{n-1}} \sigma_1 \otimes \Phi^-(\sigma'))^{\text{sm}}.$$

We get (i).

Assume that $(\pi_0)_{N_n, \eta_n} = 0$. Then N_n acts trivially on π_0 by Lemma 3.27. Hence $\pi_0|_{G_{n-1}} \hookrightarrow \Pi_{N_n}$. Applying the geometric lemma, we have $\Pi_{N_n} \cong (\text{n-Ind}_{Q_{n-1}}^{G_{n-1}} \sigma_1 \boxtimes r(\sigma'))^{\text{sm}} \otimes |\det|_F^{d/2}$ as G_{n-1} -representations. (The character $|\det|_F^{d/2}$ comes from the normalization of coinvariants.)

We also have $\pi_0|_{G_{n-1}} \hookrightarrow ((\Pi^\vee)^\vee)^{N_n} \cong ((\Pi^\vee)_{N_n})^\vee$. We calculate

$$\begin{aligned} ((\Pi^\vee)_{N_n})^\vee &\cong (((\text{n-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1^\vee \boxtimes (\sigma')^\vee)^{\text{sm}})_{N_n})^\vee \\ &\cong ((\text{n-Ind}_{Q_{n-1}}^{G_{n-1}} \sigma_1^\vee \boxtimes r((\sigma')^\vee))^{\text{sm}} \otimes |\det|_F^{d/2})^\vee \\ &\cong (\text{n-Ind}_{Q_{n-1}}^{G_{n-1}} \sigma_1 \boxtimes r((\sigma')^\vee)^\vee)^{\text{sm}} \otimes |\det|_F^{-d/2}. \end{aligned}$$

Part (ii) follows. \square

For $\pi \in \text{Rep}^\infty(P_n)$, we define the k -th derivative $\pi^{(k)} \in \text{Rep}^\infty(G_{n-k})$ of π by $\pi^{(k)} := |\det|_F^{-d/2} \otimes (\Phi^-)^{k-1}(\pi)_{N_{n-k+1}}$ for $k = 1, \dots, n$. We put $\pi^{(k)} := 0$ for $k > n$ and $\pi^{(0)} := \pi$. For $n_1, \dots, n_r \geq 1$, $\pi_{n_i} \in \text{Rep}^\infty(G_{n_i})$ ($i = 1, \dots, r$), we put $\pi_1 \times \dots \times \pi_r := (\text{n-Ind}_P^{G_{n_1+\dots+n_r}} \pi_1 \boxtimes \dots \boxtimes \pi_r)^{\text{sm}}$, where P is the standard parabolic subgroup corresponding to $n_1 + \dots + n_r$.

Lemma 3.30. *Let $\pi_1 \in \text{Rep}^\infty(G_{n_1})$ and $\pi_2 \in \text{Rep}^\infty(G_{n_2})$. Then $(\pi_1 \times \pi_2)^{(k)}$ has a filtration whose successive quotients are $\pi_1^{(l)} \times \pi_2^{(k-l)}$ with $l = 0, \dots, k$.*

Proof. The geometric lemma implies the lemma, see the proof of [BZ77, 4.5 Lemma]. \square

Lemma 3.31. *Let $\pi_0 \in \text{Rep}^\infty(P_n)$ and $k \in \mathbb{Z}_{\geq 0}$ the maximal integer such that $\pi_0^{(k)} \neq 0$. Assume that $k < n$. Then $(\Phi^-)^i(\pi_0) = 0$ for $i \geq k$.*

Proof. We prove the lemma by backward induction on i . We have $(\Phi^-)^n(\pi_0) = 0$. Assume $k \leq i < n$. By the inductive hypothesis we have $\Phi^-((\Phi^-)^i(\pi_0)) = 0$ and by $\pi_0^{(i+1)} = 0$ we have $(\Phi^-)^i(\pi_0)_{N_{n-i}} = 0$. Hence $(\Phi^-)^i(\pi_0) = 0$ by Lemma 3.27. \square

Lemma 3.32. *Assume that $\sigma_1 \not\cong \sigma_i \otimes |\text{Nrd}|_F^{-d}$ for $i = 2, \dots, n$. Then any nonzero P_n -subrepresentation π_0 of $(\text{n-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1 \boxtimes (\sigma_2 \times \dots \times \sigma_n))^{\text{sm}}$ is generic.*

Here, inside the induction, $\sigma_2 \times \dots \times \sigma_n$ is restricted from G_{n-1} to P_{n-1} .

Proof. Set $\sigma' := \sigma_2 \times \dots \times \sigma_n$. Assume that π_0 is not generic. Take the maximal k such that $\pi_0^{(k)} \neq 0$. Since π_0 is not generic, we have $k < n$. Since $(\Phi^-)^k(\pi_0) = 0$ by Lemma 3.31, $\pi_0^{(k)}$ is a subrepresentation of $(\text{n-Ind}_{Q_{n-k}}^{G_{n-k}} \sigma_1 \boxtimes (\sigma')^{(k)})^{\text{sm}}$ by Lemma 3.29. Note that $(\sigma')^{(k)}$ is of finite length by Lemma 3.30. Hence $\pi_0^{(k)}$ is also of finite length. Take an irreducible subrepresentation ω of $\pi_0^{(k)}$. By Lemma 3.30, the cuspidal support of ω is $(Z_{n-k}, \sigma_1 \boxtimes \sigma_{i_1} \boxtimes \dots \boxtimes \sigma_{i_{n-k-1}})$ for some $2 \leq i_1 < \dots < i_{n-k-1} \leq n$.

On the other hand, again by Lemma 3.29, $\pi_0^{(k)}$ is also embedded into

$$\begin{aligned} (\text{n-Ind}_{Q_{n-k}}^{G_{n-k}} \sigma_1 \boxtimes r(((\Phi^-)^{(k-1)}(\sigma'))^\vee)^\vee)^{\text{sm}} \otimes |\det|_F^{-d} \\ \cong (\text{n-Ind}_{Q_{n-k}}^{G_{n-k}} \sigma_1 \boxtimes (((\sigma')^\vee)^{(k)})^\vee)^{\text{sm}} \otimes |\det|_F^{-d}, \end{aligned}$$

where we use Lemma 3.28. Therefore the cuspidal support of ω is $(Z_{n-k}, (\sigma_1 \otimes |\text{Nrd}|_F^{-d}) \boxtimes (\sigma_{j_1} \otimes |\text{Nrd}|_F^{-d}) \boxtimes \dots \boxtimes (\sigma_{j_{n-k-1}} \otimes |\text{Nrd}|_F^{-d}))$, where $2 \leq j_1 < \dots < j_{n-k-1} \leq n$. Hence $\{\sigma_1, \sigma_{i_1}, \dots, \sigma_{i_{n-k-1}}\} = \{\sigma_1 \otimes |\text{Nrd}|_F^{-d}, \sigma_{j_1} \otimes |\text{Nrd}|_F^{-d}, \dots, \sigma_{j_{n-k-1}} \otimes |\text{Nrd}|_F^{-d}\}$. By our assumption, σ_1 is not contained in the right-hand side. Hence we have a contradiction. \square

Proof of Theorem 3.26. We prove the theorem by induction on n . For $n = 1$ there is nothing to prove, so we assume that $n \geq 2$. Set $\sigma' := \sigma_2 \times \dots \times \sigma_n$. We apply the geometric lemma to $(\text{n-Ind}_{Q_n}^{G_n} \sigma_1 \boxtimes \sigma')^{\text{sm}}|_{P_n}$. We have $Q_n \backslash G_n / P_n = \text{Aut}(\{2, \dots, n\}) \backslash S_n / \text{Aut}(\{1, \dots, n-1\}) = \{1, (1 \ 2 \ \dots \ n)\}$ and therefore we have an exact sequence

$$0 \rightarrow (\text{nc-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\text{sm}} \rightarrow (\text{n-Ind}_{Q_n}^{G_n} \sigma_1 \boxtimes \sigma')^{\text{sm}}|_{P_n} \rightarrow (\text{n-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1 \boxtimes \sigma')^{\text{sm}} \rightarrow 0.$$

We now prove that any nonzero subrepresentation $\pi_0 \subset (\text{n-Ind}_{Q_n}^{G_n} \sigma_1 \boxtimes \sigma')^{\text{sm}}|_{P_n}$ is generic. Assume that $\pi_0 \cap (\text{nc-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\text{sm}} \neq \{0\}$. By replacing π_0 with this intersection, we may assume $\pi_0 \subset (\text{nc-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\text{sm}}$. As an N_n -representation,

we have $(\mathrm{nc}\text{-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\mathrm{sm}} \cong C_c^\infty(N_n) \otimes (\sigma' \otimes \sigma_1)$. Hence there are no N_n -invariants. Therefore $\pi_0^{N_n} = \{0\}$ and this implies $(\pi_0)_{N_n, \eta_n} \neq 0$ by Lemma 3.27. We also have a P_{n-1} -isomorphism $((\mathrm{nc}\text{-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\mathrm{sm}})_{N_n, \eta_n} \cong (\sigma' \otimes \sigma_1) \otimes |\det|_F^{d/2}$, where P_{n-1} acts trivially on σ_1 . Hence by the inductive hypothesis, $(\pi_0)_{U_n, \theta} = ((\pi_0)_{N_n, \eta_n})_{U_{n-1}, \theta|_{U_{n-1}}} \neq 0$. Therefore π_0 is generic.

Next assume that $\pi_0 \cap (\mathrm{nc}\text{-Ind}_{G_{n-1}}^{P_n} \sigma' \boxtimes \sigma_1)^{\mathrm{sm}} = \{0\}$. In this case we have $\pi_0 \subset (\mathrm{n}\text{-Ind}_{Q_{n-1}N_n}^{P_n} \sigma_1 \otimes \sigma')^{\mathrm{sm}}$, and the theorem follows from Lemma 3.32. \square

3.5. Unitary case. We now state one of our main results. We need some preparations for the proof, which will be completed in subsection 3.9. For $\alpha \in \Phi_{\mathrm{red}}^+$, let L'_α be the (closed) subgroup generated by $U \cap L_\alpha$ and $\bar{U} \cap L_\alpha$, where \bar{U} is the unipotent radical of the parabolic subgroup opposite to \underline{B} . It has a concrete description, cf. [AHHV17, II.4].

For a character $\omega: G \rightarrow C^\times$ we define $e(\omega) = e_G(\omega) \in \mathfrak{a}_{G, \mathbb{R}}^*$ by $|\omega|_C = \chi_{e(\omega)}$, where the absolute value $|\cdot|_C$ on C is normalized such that $|p|_C = p^{-1}$. (Recall that the unramified character χ_ν for $\nu \in \mathfrak{a}_{G, \mathbb{C}}^*$ was defined in §3.1.) In particular, it follows that $e(\omega) = [F : \mathbb{Q}_p]^{-1}\omega$ if $\omega \in X^*(G) \subset \mathfrak{a}_{G, \mathbb{R}}^*$, as $|\cdot|_C^{[F : \mathbb{Q}_p]}$ restricts to $|\cdot|_F$ on $F \subset C$.

Theorem 3.33. *Assume Assumption 2.14. Let σ be a finite-dimensional absolutely irreducible continuous representation of L and ω_σ the central character of σ .*

- (i) *The action of Z on the coinvariants $\sigma_{U \cap L}$ is absolutely irreducible, so it has a central character $\omega_{\sigma_{U \cap L}}$.*
- (ii) *If $e(\omega_\sigma|_{A_L})$ is dominant, then $(\mathrm{Ind}_B^G \sigma)^{\mathrm{cts}}$ is reducible if and only if, after perhaps replacing C by a finite extension, $\underline{L}(\sigma') \in \mathcal{O}^{\tilde{P}_1}$ for a parabolic subgroup $\underline{P}_1 \supseteq \underline{P}$. Here, $\tilde{P}_1 := \mathrm{Res}_{F/\mathbb{Q}_p} \underline{P}_1$.*
- (iii) *If $e(\omega_{\sigma_{U \cap L}}|_S)$ is dominant, then $(\mathrm{Ind}_P^G \sigma)^{\mathrm{cts}}$ is reducible if and only if there exists $\alpha \in \Delta \setminus \Delta_L$ such that $\sigma|_{Z \cap L'_\alpha}$ is trivial.*

Recall that $\sigma' \in \mathcal{O}^{\tilde{L}}$ after perhaps replacing C by a finite extension (Lemma 2.49), where $\tilde{L} := \mathrm{Res}_{F/\mathbb{Q}_p} \underline{L}$. Note that $e(\omega_\sigma|_{A_L}) = e(\omega_{\sigma_{U \cap L}}|_{A_L})$ is dominant if $e(\omega_{\sigma_{U \cap L}}|_S)$ is dominant by Lemma 3.2(ii). By [AHHV17, II.7 Proposition] and Lemma 2.4, $\sigma|_{Z \cap L'_\alpha}$ is trivial for some $\alpha \in \Delta \setminus \Delta_L$ if and only if σ extends to a continuous representation of a larger Levi subgroup. The condition $\underline{L}(\sigma') \in \mathcal{O}^{\tilde{P}_1}$ in (ii) is made more explicit in Lemma 2.29.

Remark 3.34. We say that a Banach representation σ of L is *unitarizable* if it admits an L -invariant defining norm or equivalently an L -invariant open and bounded \mathcal{O}_C -lattice. Suppose now that σ has a central character ω_σ (for example, if it is admissible and absolutely irreducible). Then σ unitarizable implies that ω_σ is unitary, or equivalently $e(\omega_\sigma|_{A_L}) = 0$. If $L = Z$, then the converse is true too. (To see that if ω_σ is unitary then σ is unitarizable, note that $Z_0 S$ is of finite index in Z , where Z_0 is the unique maximal compact subgroup of Z . By [Eme17, Lemma 6.5.5] there exists a bounded open Z_0 -invariant lattice Λ in σ , so $\sum_{z \in Z/Z_0 S} z\Lambda$ is a bounded open Z -invariant lattice.) If σ is unitarizable then $\sigma_{U \cap L}$ is also unitarizable. Hence the hypotheses on central characters in parts (ii) and (iii) of Theorem 3.33 hold for any unitarizable σ .

Remark 3.35. Schneider [Sch06, Conjecture 2.5] stated a conjecture concerning the irreducibility of Banach principal series of a split simply-connected group over a p -adic field, under an antidominance condition on the inducing character, as discussed in the introduction. Theorem 3.33 resolves this completely for unitarizable inducing representations, for an essentially arbitrary connected reductive group.

Remark 3.36. When \underline{G} is split, $\underline{P} = \underline{B}$ and $e(\omega_\sigma|_S)$ is strictly dominant, i.e. $\langle e(\omega_\sigma|_S), \alpha^\vee \rangle > 0$ for any positive root α , Theorem 3.33 was proved by Ban–Hundley [BH16] (at least when the derived subgroup is simply connected). Note that in this case there are no reducibilities, as $\sigma|_{Z \cap L'_\alpha}$ trivial implies $\langle e(\omega_\sigma|_S), \alpha^\vee \rangle = 0$.

Remark 3.37. We note that irreducible admissible Banach representations of Z do not have to be finite-dimensional, even though \underline{Z} is anisotropic modulo its center. For example, when $Z = D^\times$ with D a central division algebra, [Paš22] shows the existence of an irreducible admissible representation of δ -dimension 1 (in particular, of infinite dimension).

3.6. Rationality: the quasisplit case. To prove Theorem 3.33, we require the rationality of reducibility points of certain smooth parabolic inductions over \mathbb{C} . We will finish the proof of the necessary property in subsection 3.8. *In subsections 3.6–3.8 all representations are smooth and over \mathbb{C} .*

Suppose now that \underline{G} is quasisplit, so that \underline{Z} is a maximal torus. For $\nu \in \mathfrak{a}_{\underline{Z}, \mathbb{C}}^*$ and $\alpha \in \Phi(\underline{G}, \underline{A}_{\underline{Z}})$ we can define $\langle \nu, \alpha^\vee \rangle_{\text{abs}} := \langle i(\nu), \tilde{\alpha}^\vee \rangle$, where $i: \mathfrak{a}_{\underline{Z}, \mathbb{C}}^* = X^*(\underline{Z}) \otimes \mathbb{C} \hookrightarrow X^*(\underline{Z} \times \overline{F}) \otimes \mathbb{C}$ is the natural map and $\tilde{\alpha} \in X^*(\underline{Z} \times \overline{F})$ is any absolute root lifting α . Note that this is well defined because the lifts $\tilde{\alpha}$ form a $\text{Gal}(\overline{F}/F)$ -orbit. Note that $\langle \alpha, \alpha^\vee \rangle_{\text{abs}} \in \mathbb{Q}_{>0}$. (In fact, if d denotes the number of lifts $\tilde{\alpha}$ of α we have $\langle \nu, \alpha^\vee \rangle_{\text{abs}} = \frac{1}{2d} \langle \nu, \alpha^\vee \rangle$ if 2α is a root and $\langle \nu, \alpha^\vee \rangle_{\text{abs}} = \frac{1}{d} \langle \nu, \alpha^\vee \rangle$ otherwise, cf. [AHV22, Lemma A.9].)

Lemma 3.38. *Suppose that \underline{G} is quasisplit, $\underline{P} = \underline{L}\underline{N}$ maximal and semistandard, σ a generic supercuspidal representation of L . If $\mu^G(\sigma\chi_\nu)$ has a pole at $\nu = \nu_0 \in \mathfrak{a}_{L, \mathbb{R}}^*$, then $\nu_0 \in (\mathfrak{a}_{L, \mathbb{Q}}^G)^* \oplus \mathfrak{a}_{G, \mathbb{R}}^*$. Equivalently, if $\mu^G(\sigma\delta_P^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$. More precisely, there exists $\gamma \in \Phi(\underline{G}, \underline{A}_{\underline{Z}})$ occurring in \underline{N} such that $\langle \nu, \gamma^\vee \rangle_{\text{abs}} \in \{\pm\frac{1}{2}, \pm 1\}$.*

Proof. First note that χ_ν extends to an unramified character of G if and only if $\nu \in \mathfrak{a}_{G, \mathbb{R}}^*$, so we may assume $\nu \in (\mathfrak{a}_{L, \mathbb{R}}^G)^*$. As P is maximal we can write $\Delta(\underline{P}, \underline{A}_{\underline{L}}) = \{\alpha\}$, so $2\rho_P \in c\alpha$ for some $c \in \mathbb{Z}_{>0}$, i.e. $\delta_P = \chi_{c\alpha}$. Moreover, α is a basis of $(\mathfrak{a}_{L, \mathbb{Q}}^G)^*$, which explains the equivalence of the two statements. Choose a Borel subgroup \underline{B} contained in \underline{P} . Then there is a unique simple root $\gamma \in \Delta(\underline{B}, \underline{A}_{\underline{Z}})$ that occurs in \underline{N} , and [Sha90, Theorem 8.1] implies that $\langle \nu, \gamma^\vee \rangle_{\text{abs}} \in \{\pm\frac{1}{2}, \pm 1\}$. As $\langle \alpha, \alpha^\vee \rangle_{\text{abs}} \in \mathbb{Q}_{>0}$, we deduce the second statement. \square

Proposition 3.39. *Suppose that \underline{G} is quasisplit, τ a discrete series representation of G whose supercuspidal support is generic. Then there exists a standard parabolic subgroup $\underline{P} = \underline{L}\underline{N}$, a unitary (generic) supercuspidal representation σ of L and $\nu \in (\mathfrak{a}_{L, \mathbb{Q}}^G)^* \subset \mathfrak{a}_{L, \mathbb{R}}^*$ such that τ is a quotient of $(\text{n-Ind}_P^G \sigma\chi_\nu)^{\text{sm}}$. More precisely, $\nu = \sum_{\alpha \in \Delta(\underline{P}, \underline{A}_{\underline{L}})} c_\alpha \alpha$, where $c_\alpha \in \mathbb{Q}_{<0}$ for all α .*

Proof. All except the rationality of ν , or equivalently of the c_α , follows from [Sil80b, p. 582] or [Wal03, Proposition III.1.1]. (The parabolic subgroup there may be conjugated to be standard.) In particular, $\nu \in (\mathfrak{a}_{L, \mathbb{R}}^G)^*$. By [Sil80b, p. 582] there exists a linearly independent set of reduced roots $\alpha_1, \dots, \alpha_\ell \in \Phi_{\text{red}}(\underline{P}, \underline{A}_{\underline{L}})$ with $\ell = |\Delta(\underline{P}, \underline{A}_{\underline{L}})|$ such that $\mu^{L\alpha_i}(\sigma\chi_{\nu'})$ has a pole at $\nu' = \nu$ for $1 \leq i \leq \ell$. By Lemma 3.38 we deduce that the projection of ν under the projection $(\mathfrak{a}_{L, \mathbb{R}}^G)^* \rightarrow (\mathfrak{a}_{L, \mathbb{R}}^{L\alpha_i})^*$ lies in $(\mathfrak{a}_{L, \mathbb{Q}}^{L\alpha_i})^*$ for all i . It therefore suffices to show that the natural map $(\mathfrak{a}_{L, \mathbb{R}}^G)^* \rightarrow \bigoplus_{i=1}^\ell (\mathfrak{a}_{L, \mathbb{R}}^{L\alpha_i})^*$ is an isomorphism, or equivalently injective, or equivalently that $\bigcap_i \mathfrak{a}_{L\alpha_i, \mathbb{R}}^* = \mathfrak{a}_{G, \mathbb{R}}^*$.

By dimension reasons we see that $\mathfrak{a}_{L\alpha_i, \mathbb{R}}^*$ is the annihilator of α_i^\vee in $\mathfrak{a}_{L, \mathbb{R}}^*$. By Lemma 3.1, the coroots $\alpha_1^\vee, \dots, \alpha_\ell^\vee$ are linearly independent in $\mathfrak{a}_{L, \mathbb{R}}$, hence a basis of $\mathfrak{a}_{L, \mathbb{R}}^G$, so indeed $\bigcap_i \mathfrak{a}_{L\alpha_i, \mathbb{R}}^* \subset \mathfrak{a}_{G, \mathbb{R}}^*$. \square

Remark 3.40. Continue to suppose the hypotheses in Proposition 3.39. If \underline{G} is a subgroup of $\underline{G}' := (\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i})/\underline{H}$ containing $(\underline{G}')^{\text{der}}$, for some finite extensions E_i/F , integers $n_i \geq 1$ and a central subtorus \underline{H} of $\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i}$ that is moreover induced, then $\langle \nu, \alpha^\vee \rangle_{\text{abs}} \in \mathbb{Z}$ for all $\alpha \in \Phi(\underline{G}, \underline{A}_{\underline{Z}})$. (First reduce to \underline{G}' ; then reduce to $\underline{H} = 1$; finally check it in the case of $\text{Res}_{E/F} \text{GL}_n$ using [Zel80, Theorem 9.3].)

Proposition 3.41. *Suppose that \underline{G} is quasisplit, $\underline{Q} = \underline{L}_{\underline{Q}}\underline{N}_{\underline{Q}}$ a standard parabolic subgroup, and τ a discrete series representation of $L_{\underline{Q}}$ whose supercuspidal support is generic. If $\mu^G(\tau\delta_{\underline{Q}}^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$.*

Proof. By Proposition 3.39 we can write τ as quotient of $(\text{n-Ind}_{P \cap L_{\underline{Q}}}^{L_{\underline{Q}}} \sigma\chi_\nu)^{\text{sm}}$ for some standard parabolic $\underline{P} = \underline{L}\underline{N}$ contained in \underline{Q} , some unitary supercuspidal representation σ of L , and some $\nu \in (\mathfrak{a}_{L, \mathbb{Q}}^{L_{\underline{Q}}})^*$. Write $\delta_{\underline{Q}} = \chi_{\nu'}$ for $\nu' = 2\rho_{\underline{Q}} \in (\mathfrak{a}_{L_{\underline{Q}}, \mathbb{Q}}^G)^*$. Hence $(\text{n-Ind}_{P \cap L_{\underline{Q}}}^{L_{\underline{Q}}} \sigma\chi_{\nu+s\nu'})^{\text{sm}}$ surjects onto $\tau\delta_{\underline{Q}}^s$. By [Sil80a, Theorem 1] and Proposition 3.4 we have

$$\mu^G(\tau\delta_{\underline{Q}}^s) = \frac{\mu^G(\sigma\chi_{\nu+s\nu'})}{\mu^{L_{\underline{Q}}}(\sigma\chi_{\nu+s\nu'})} = \prod_{\beta \in \Phi_{\text{red}}(\underline{P}, \underline{A}_{\underline{L}}) \setminus \Phi_{\text{red}}(\underline{P} \cap \underline{L}_{\underline{Q}}, \underline{A}_{\underline{L}})} \mu^{L_\beta}(\sigma\chi_{\nu+s\nu'}).$$

If $\mu^G(\tau\delta_{\underline{Q}}^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then by Lemma 3.38 we deduce that there exists $\beta \in \Phi_{\text{red}}(\underline{P}, \underline{A}_{\underline{L}}) \setminus \Phi_{\text{red}}(\underline{P} \cap \underline{L}_{\underline{Q}}, \underline{A}_{\underline{L}})$ such that $\nu + s_0\nu' \in (\mathfrak{a}_{L, \mathbb{Q}}^{L_\beta})^* \oplus \mathfrak{a}_{L_\beta, \mathbb{R}}^*$ or equivalently $s_0\nu' \in (\mathfrak{a}_{L, \mathbb{Q}}^{L_\beta})^* \oplus \mathfrak{a}_{L_\beta, \mathbb{R}}^*$ (by rationality of ν).

We claim that $\nu' \notin \mathfrak{a}_{L_\beta, \mathbb{R}}^*$, i.e. $\langle \nu', \beta^\vee \rangle_L \neq 0$, where $\langle \cdot, \cdot \rangle_L$ denotes the pairing $\mathfrak{a}_{L, \mathbb{R}}^* \times \mathfrak{a}_{L, \mathbb{R}} \rightarrow \mathbb{R}$. Note that $\beta_{\underline{Q}} := \beta|_{\underline{A}_{\underline{L}_{\underline{Q}}}} \neq 1$, i.e. $\beta_{\underline{Q}} \in \Phi(\underline{G}, \underline{A}_{\underline{L}_{\underline{Q}}})$. (If not, then $\underline{A}_{\underline{L}_{\underline{Q}}}$ is contained in $\ker(\beta)$ and so $\underline{A}_{\underline{L}_{\underline{Q}}}$ centralizes the unipotent subgroup $\underline{U}_{(\beta)} \subset \underline{P}$ whose Lie algebra consists of positive integer multiples of β , i.e. $\underline{U}_{(\beta)} \subset \underline{L}_{\underline{Q}}$, which is a contradiction.) Hence $\langle \nu', \beta^\vee \rangle_L = \langle \nu', \beta_{\underline{Q}}^\vee \rangle_{L_{\underline{Q}}} \neq 0$ by Lemma 3.1(i), as $\nu' = 2\rho_{\underline{Q}}$ in $(\mathfrak{a}_{L_{\underline{Q}}, \mathbb{Q}}^G)^*$, proving the claim. Therefore, the image of ν' under the projection $\mathfrak{a}_{L, \mathbb{Q}}^* \twoheadrightarrow (\mathfrak{a}_{L, \mathbb{Q}}^{L_\beta})^*$ has to be nonzero, and hence $s_0 \in \mathbb{Q}$. \square

Remark 3.42. More precisely, the above proof shows that (in the notation of the proof) there exists a root $\gamma \in \Phi(\underline{G}, \underline{A}_{\underline{Z}})$ that occurs in $\underline{N}_{\underline{Q}}$ such that $\langle \nu + 2s\rho_{\underline{Q}}, \gamma^\vee \rangle_{\text{abs}} \in \{\pm\frac{1}{2}, \pm 1\}$. In particular, in case the Levi subgroup $\underline{L}_{\underline{Q}}$ satisfies the assumption of Remark 3.40 we deduce that $s\langle 2\rho_{\underline{Q}}, \gamma^\vee \rangle_{\text{abs}} \in \frac{1}{2}\mathbb{Z}$.

3.7. Rationality: inner forms. In this subsection we prove the rationality of the reducibility points of $(\text{n-Ind}_P^G \sigma\delta_P^s)^{\text{sm}}$ for some inner forms \underline{G} , which include all simply-connected almost simple groups of rank one (cf. subsection 3.8 below). For hermitian quaternionic groups of maximal Witt rank Muić–Savin [MS00, §2] proved that the μ -function of a discrete series representation on a Siegel Levi coincides with the μ -function of its Jacquet–Langlands transfer by using a global argument. This method was used in some further cases in [Kon03] and [Cho14]. We will prove a similar result in greater generality, though we do not determine the μ -function completely. We remind the reader that in this subsection all representations are smooth and complex.

Our setting is the following: \underline{G} is any connected reductive group, $\underline{P} = \underline{L}\underline{N}$ is a maximal parabolic subgroup such that \underline{L} is an inner form of a group \underline{L}' satisfying

$$(\tilde{\underline{L}}')^{\text{der}} \subset \underline{L}' \subset \tilde{\underline{L}}',$$

where

$$\tilde{\underline{L}}' := \left(\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i} \right) / \underline{H}$$

for some finite extensions E_i/F , integers $n_i \geq 1$ and a central subtorus \underline{H} of $\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i}$. We moreover assume that \underline{H} is an induced torus. Then we prove the following.

Theorem 3.43. *Let σ be a discrete series representation of L . If $\mu^G(\sigma\delta_P^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$.*

We obtain the following corollary from our argument. The inner form \underline{L} of \underline{L}' gives rise to an inner form $\tilde{\underline{L}}$ of $\tilde{\underline{L}}'$ such that $\tilde{\underline{L}}^{\text{der}} \subset \underline{L} \subset \tilde{\underline{L}}$, as we now explain. We have $\underline{Z}_{\underline{L}'} = \underline{L}' \cap \underline{Z}_{\tilde{\underline{L}}'}$ and hence a map $\underline{L}'/\underline{Z}_{\underline{L}'} \rightarrow \tilde{\underline{L}}'/\underline{Z}_{\tilde{\underline{L}}'}$ that induces $H^1(F, \underline{L}'/\underline{Z}_{\underline{L}'}) \rightarrow H^1(F, \tilde{\underline{L}}'/\underline{Z}_{\tilde{\underline{L}}'})$. The inner form \underline{L} of \underline{L}' corresponds to an element of $H^1(F, \underline{L}'/\underline{Z}_{\underline{L}'})$ and by the above map we get the desired inner form $\tilde{\underline{L}}$ of $\tilde{\underline{L}}'$.

Corollary 3.44. *Suppose that σ_1, σ_2 are discrete series representations of L that are conjugate under the action of \tilde{L} . Then $\mu^G(\sigma_1\delta_P^s) = \mu^G(\sigma_2\delta_P^s)$.*

This verifies [Cho14, Working Hypothesis 1.1] (in our more general setting). See also Remark 3.54.

We use a global method, and we start by giving some notation for real and global groups. Bold letters will be used for global objects: for example, \mathbf{F} will be a number field and \mathbf{G} will be a connected reductive group over \mathbf{F} . For each place v of \mathbf{F} , \mathbf{F}_v denotes the completion of \mathbf{F} at v . If v is an infinite place of \mathbf{F} , a representation of $\mathbf{G}(\mathbf{F}_v)$ means a $(\text{Lie}(\mathbf{G}(\mathbf{F}_v)) \otimes_{\mathbb{R}} \mathbb{C}, K_v)$ -module, where K_v is a fixed maximal compact subgroup of $\mathbf{G}(\mathbf{F}_v)$. If $\mathbf{P} = \mathbf{L}\mathbf{N}$ is a parabolic subgroup of \mathbf{G} and σ a representation of $\mathbf{G}(\mathbf{F}_v)$, let $(\text{n-Ind}_{\mathbf{P}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \sigma)^{\text{sm}}$ be the normalized parabolic induction in the context of $(\text{Lie}(\mathbf{G}(\mathbf{F}_v)) \otimes_{\mathbb{R}} \mathbb{C}, K_v)$ -modules. Let $\mathbb{A} = \mathbb{A}_{\mathbf{F}}$ be the adèle ring of \mathbf{F} . Let $(\text{n-Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} \sigma)^{\text{sm}}$ be the normalized parabolic induction for a cuspidal automorphic representation σ of $L(\mathbb{A})$ (cf. [MW95, II.1]).

Lemma 3.45. *There exists a number field \mathbf{F} , a place v_0 of \mathbf{F} , and finite extensions $\mathbf{E}/\mathbf{F}, \mathbf{E}_1/\mathbf{F}, \dots, \mathbf{E}_r/\mathbf{F}$ such that \mathbf{E}/\mathbf{F} is Galois, $\mathbf{F}_{v_0} \cong \mathbf{F}$, v_0 does not decompose in \mathbf{E} , \underline{G} splits over \mathbf{E}_{v_0} , $\mathbf{E} \supset \mathbf{E}_i$, and $(\mathbf{E}_i)_{v_0} \cong \mathbf{E}_i$. In particular we have $\text{Gal}(\mathbf{E}_{v_0}/\mathbf{F}) \cong \text{Gal}(\mathbf{E}/\mathbf{F})$.*

Proof. Take a finite Galois extension E/F such that \underline{G} splits over E and $E \supset E_i$ for each $i = 1, \dots, r$. Then by [GL18, Lemma 3.1] there exists a Galois extension \mathbf{E}/\mathbf{F} of number fields and a place v_0 of \mathbf{F} such that v_0 does not decompose in \mathbf{E} , $\mathbf{E}_{v_0} \cong E$, $\mathbf{F}_{v_0} \cong F$. The subfield $E_i \subset E$ corresponds to a subgroup $H_i \subset \text{Gal}(E/F) \cong \text{Gal}(\mathbf{E}/\mathbf{F})$ and let \mathbf{E}_i be the subfield of \mathbf{E} corresponding to H_i . Then this gives the desired properties. \square

Let \underline{G}' be the quasisplit inner form of \underline{G} , and fix an inner twist $\xi: \underline{G}'_{\overline{\mathbf{F}}} \xrightarrow{\sim} \underline{G}'_{\overline{\mathbf{F}}}$ over $\overline{\mathbf{F}}$. After conjugating by $\underline{G}'(\overline{\mathbf{F}})$ we may assume that the parabolic subgroup $\xi(\underline{P}_{\overline{\mathbf{F}}})$ and its Levi $\xi(\underline{L}_{\overline{\mathbf{F}}})$ are defined over F ([Sol20, Lemma 3]), and we let $\underline{P}' = \underline{L}'\underline{N}'$ the parabolic subgroup of \underline{G}' obtained in this way, i.e. $\xi(\underline{P}_{\overline{\mathbf{F}}}) = \underline{P}'_{\overline{\mathbf{F}}}$ and $\xi(\underline{L}_{\overline{\mathbf{F}}}) = \underline{L}'_{\overline{\mathbf{F}}}$. Then ξ induces an inner twist $\underline{L}_{\overline{\mathbf{F}}} \xrightarrow{\sim} \underline{L}'_{\overline{\mathbf{F}}}$ ([Sol20, Lemma 3]) and so \underline{L}' is the

unique quasisplit inner form, consistent with our notation above. In particular, ξ induces F -isomorphisms $\underline{Z}_{\underline{L}} \cong \underline{Z}_{\underline{L}'}$ and $\underline{A}_{\underline{L}} \cong \underline{A}_{\underline{L}'}$. We note that \underline{G}' splits over \mathbf{E}_{v_0} , because \underline{L}' splits over \mathbf{E}_{v_0} . (Note that $(\underline{L}')^{\text{der}} = (\tilde{\underline{L}}')^{\text{der}}$ splits over \mathbf{E}_{v_0} and similarly for the radical $\underline{Z}_{\underline{L}'}$, noting that $\underline{Z}_{\underline{L}'} = \underline{L}' \cap \underline{Z}_{\tilde{\underline{L}}'}$.)

The quasisplit group \underline{G}' is classified by the action of $\text{Gal}(\mathbf{E}_{v_0}/F)$ on the root datum of \underline{G}' . Since $\text{Gal}(\mathbf{E}_{v_0}/F) \cong \text{Gal}(\mathbf{E}/F)$, it also determines a quasisplit group \mathbf{G}' over \mathbf{F} . The parabolic subgroup \underline{P}' corresponds to a set of simple roots and let $\mathbf{P}' = \mathbf{L}'\mathbf{N}' \subset \mathbf{G}'$ be the corresponding parabolic subgroup. Then we have $(\mathbf{G}'_{v_0}, \mathbf{L}'_{v_0}, \mathbf{N}'_{v_0}) \cong (\underline{G}', \underline{L}', \underline{N}')$, where $\mathbf{G}'_{v_0} := \mathbf{G}' \times_{\mathbf{F}} \mathbf{F}_{v_0}$ (and this notation is used for other groups as well).

Lemma 3.46. *There exists an inner form $(\mathbf{G}, \mathbf{P}, \mathbf{L})$ of $(\mathbf{G}', \mathbf{P}', \mathbf{L}')$ such that $(\mathbf{G}_{v_0}, \mathbf{P}_{v_0}, \mathbf{L}_{v_0}) \cong (\underline{G}, \underline{P}, \underline{L})$ and $(\mathbf{G}_v, \mathbf{P}_v, \mathbf{L}_v) \cong (\mathbf{G}'_v, \mathbf{P}'_v, \mathbf{L}'_v)$ for any infinite place v .*

Proof. This follows like in the proof of [GL18, Lemma 3.2]. Let V_{∞} be the set of infinite places. Inner forms of $(\mathbf{G}', \mathbf{P}', \mathbf{L}')$ are classified by $H^1(\mathbf{F}, I(\mathbf{G}', \mathbf{P}', \mathbf{L}'))$, where $I(\mathbf{G}', \mathbf{P}', \mathbf{L}')$ is the group of inner automorphisms of the triple $(\mathbf{G}', \mathbf{P}', \mathbf{L}')$. Since the normalizer of \mathbf{P}' in \mathbf{G}' is \mathbf{P}' and the normalizer of \mathbf{L}' in \mathbf{P}' is \mathbf{L}' , we have $I(\mathbf{G}', \mathbf{P}', \mathbf{L}') \cong \mathbf{L}'/\mathbf{Z}_{\mathbf{G}'}$, where $\mathbf{Z}_{\mathbf{G}'}$ is the center of \mathbf{G}' . Therefore it is sufficient to prove that the map

$$(3.6) \quad H^1(\mathbf{F}, \mathbf{L}'/\mathbf{Z}_{\mathbf{G}'}) \rightarrow \prod_{v \in V_{\infty} \cup \{v_0\}} H^1(\mathbf{F}_v, \mathbf{L}'/\mathbf{Z}_{\mathbf{G}'})$$

is surjective. By replacing \mathbf{G}' with $\mathbf{G}'/\mathbf{Z}_{\mathbf{G}'}$, we assume that $\mathbf{Z}_{\mathbf{G}'}$ is trivial. Hence it is sufficient to prove that $H^1(\mathbf{F}, \mathbf{L}') \rightarrow \prod_{v \in V_{\infty} \cup \{v_0\}} H^1(\mathbf{F}_v, \mathbf{L}')$ is surjective.

First we observe that $\mathbf{Z}_{\mathbf{L}'}$ is an induced torus. Indeed, let $\overline{\mathbf{F}}$ be an algebraic closure of \mathbf{F} . Since $\mathbf{Z}_{\mathbf{G}'}$ is trivial, $\mathbf{Z}_{\mathbf{L}'}$ is a torus and the fundamental coweights form a basis of the cocharacter lattice $X_*(\mathbf{T}' \times_{\mathbf{F}} \overline{\mathbf{F}})$, where $\mathbf{T}' \subset \mathbf{L}'$ is a minimal Levi subgroup (a torus). Then a Galois-stable subset of fundamental coweights is a basis of $X_*(\mathbf{Z}_{\mathbf{L}'} \times_{\mathbf{F}} \overline{\mathbf{F}})$, i.e. $\mathbf{Z}_{\mathbf{L}'}$ is an induced torus. Hence there exist finite extensions $\mathbf{E}_1, \dots, \mathbf{E}_r$ such that $\mathbf{Z}_{\mathbf{L}'} \cong \prod_i \text{Res}_{\mathbf{E}_i/\mathbf{F}} \mathbb{G}_m$. Let $v' \neq v_0$ be a finite place of \mathbf{F} which does not decompose in \mathbf{E}_i for $i = 1, \dots, r$ and we regard v' also as a place of \mathbf{E}_i . Let $V(\mathbf{F})$ (resp. $V(\mathbf{E}_i)$) be the set of places of \mathbf{F} (resp. \mathbf{E}_i). Then

$$\begin{array}{ccc} H^2(\mathbf{F}, \mathbf{Z}_{\mathbf{L}'}) & \longrightarrow & \bigoplus_{v \in V(\mathbf{F}) \setminus \{v'\}} H^2(\mathbf{F}_v, \mathbf{Z}_{\mathbf{L}'}) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{i=1}^r H^2(\mathbf{E}_i, \mathbb{G}_m) & \longrightarrow & \bigoplus_{i=1}^r \bigoplus_{v \in V(\mathbf{E}_i) \setminus \{v'\}} H^2((\mathbf{E}_i)_v, \mathbb{G}_m). \end{array}$$

By the local-global principle for Brauer groups we see as in [PR06, Lemma 3] that the bottom horizontal map is an isomorphism.

Set $V' := V(\mathbf{F}) \setminus \{v'\}$. As $\mathbf{Z}_{\mathbf{L}',v}$ is an induced torus, for any $v \in V(\mathbf{F})$, $H^1(\mathbf{F}_v, \mathbf{Z}_{\mathbf{L}'})$ is trivial by Hilbert 90. Hence $H^1(\mathbf{F}_v, \mathbf{L}') \rightarrow H^1(\mathbf{F}_v, \mathbf{L}'/\mathbf{Z}_{\mathbf{L}'})$ is injective [Ser02, I.5.7 Proposition 42]. We have the commutative diagram of pointed sets with exact rows:

$$\begin{array}{ccccc} H^1(\mathbf{F}, \mathbf{L}') & \longrightarrow & H^1(\mathbf{F}, \mathbf{L}'/\mathbf{Z}_{\mathbf{L}'}) & \longrightarrow & H^2(\mathbf{F}, \mathbf{Z}_{\mathbf{L}'}) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{v \in V'} H^1(\mathbf{F}_v, \mathbf{L}') & \hookrightarrow & \bigoplus_{v \in V'} H^1(\mathbf{F}_v, \mathbf{L}'/\mathbf{Z}_{\mathbf{L}'}) & \longrightarrow & \bigoplus_{v \in V'} H^2(\mathbf{F}_v, \mathbf{Z}_{\mathbf{L}'}) \end{array}$$

Since $\mathbf{Z}_{\mathbf{L}'}$ is an induced torus, the right vertical map is an isomorphism [PR06, Lemma 3]. We also have that the middle map is surjective [PR06, Proposition 1]. Therefore the left vertical map is surjective and hence (3.6) is surjective. \square

Take \mathbf{G} and $\mathbf{P} = \mathbf{LN}$ as in the lemma. We fix an inner twist $\xi: \mathbf{G}_{\overline{\mathbf{F}}} \xrightarrow{\sim} \mathbf{G}'_{\overline{\mathbf{F}}}$ sending $(\mathbf{P}_{\overline{\mathbf{F}}}, \mathbf{L}_{\overline{\mathbf{F}}})$ to $(\mathbf{P}'_{\overline{\mathbf{F}}}, \mathbf{L}'_{\overline{\mathbf{F}}})$ such that ξ_{v_0} is equivalent to ξ and ξ_v is trivial for all infinite places v . As above, ξ induces an isomorphism $\mathbf{A}_{\mathbf{L}} \cong \mathbf{A}_{\mathbf{L}'}$ which becomes the above isomorphism $\underline{\mathbf{A}}_{\mathbf{L}} \cong \underline{\mathbf{A}}_{\mathbf{L}'}$ after base change to $\mathbf{F}_{v_0} = F$.

Recall that we have $\underline{\mathbf{H}} \subset \prod_{i=1}^r \text{Res}_{\mathbf{E}_i/F} \text{GL}_1$. Such a subtorus is classified by a $\text{Gal}(\mathbf{E}_{v_0}/F)$ -stable saturated subgroup of $X^*(\prod_{i=1}^r \text{Res}_{\mathbf{E}_i/F} \text{GL}_1 \times_F \mathbf{E}_{v_0})$ and therefore by Lemma 3.45 we have a subtorus $\mathbf{H} \subset \prod_{i=1}^r \text{Res}_{\mathbf{E}_i/F} \text{GL}_1$ such that $\mathbf{H}_{v_0} \cong \underline{\mathbf{H}}$. As $\underline{\mathbf{H}}$ is an induced torus, so is \mathbf{H} . We put $\tilde{\mathbf{L}}' := (\prod_{i=1}^r \text{Res}_{\mathbf{E}_i/F} \text{GL}_{n_i})/\mathbf{H}$. The subgroups $(\tilde{\mathbf{L}}')^{\text{der}} \subset \mathbf{L}'' \subset \tilde{\mathbf{L}}'$ are in bijection with subgroups $(\tilde{\mathbf{L}}')^{\text{der}} \subset \underline{\mathbf{L}}'' \subset \tilde{\mathbf{L}}'$ upon completion at v_0 (since $\tilde{\mathbf{L}}'/(\tilde{\mathbf{L}}')^{\text{der}}$ is a torus that splits over \mathbf{E}), and any such \mathbf{L}'' is quasisplit and splits over \mathbf{E} . In particular, we get $(\tilde{\mathbf{L}}')^{\text{der}} \subset \mathbf{L}' \subset \tilde{\mathbf{L}}'$.

Let $\mathbf{Z}_{\mathbf{L}'}$ (resp. $\mathbf{Z}_{\tilde{\mathbf{L}}'}$) be the center of \mathbf{L}' (resp. $\tilde{\mathbf{L}}'$). As explained before Corollary 3.44, the inner form \mathbf{L} of \mathbf{L}' gives rise to an inner form $\tilde{\mathbf{L}}$ of $\tilde{\mathbf{L}}'$ such that $\tilde{\mathbf{L}}^{\text{der}} \subset \mathbf{L} \subset \tilde{\mathbf{L}}$. We remark that $\tilde{\mathbf{L}}_{v_0} \cong \underline{\tilde{\mathbf{L}}}$ and that $\mathbf{Z}_{\tilde{\mathbf{L}}} = \mathbf{Z}_{\tilde{\mathbf{L}'}}$ (a torus).

Recall that we have a discrete series representation σ of $L \cong \mathbf{L}(\mathbf{F}_{v_0})$. Take an irreducible discrete representation $\tilde{\sigma}$ of $\tilde{\mathbf{L}}(\mathbf{F}_{v_0})$ whose restriction to L contains σ . (See for example [Tad92, §2].)

Recall that $\mathbb{A} = \mathbb{A}_{\mathbf{F}}$ is the adèle ring of \mathbf{F} .

Lemma 3.47. *There exists a unitary character $\mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ whose v_0 -component is the central character of $\tilde{\sigma}$.*

Proof. First we remark that the \mathbf{F} -rational rank of $\mathbf{Z}_{\tilde{\mathbf{L}'}}$ is equal to the F -rational rank of $\mathbf{Z}_{\tilde{\mathbf{L}'}, v_0}$ by Lemma 3.45.

Let $\omega_{\tilde{\sigma}}$ be the central character of $\tilde{\sigma}$ and V^∞ be the set of finite places of \mathbf{F} . For each $v \in V^\infty$, let $\mathbf{Z}_{\tilde{\mathbf{L}'}, v}^0$ be the maximal compact subgroup of $\mathbf{Z}_{\tilde{\mathbf{L}'}, v}$ (which is the unique parahoric subgroup for almost all v). Then $K := \prod_{v \in V^\infty} \mathbf{Z}_{\tilde{\mathbf{L}'}, v}^0$ is a compact subgroup of $\mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A})$. Note that $\mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \cap K$ is trivial since there is an infinite place. Hence $K \hookrightarrow \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A})$ is a closed subgroup. Consider the character of K defined by $(k_v)_{v \in V^\infty} \mapsto \omega_{\tilde{\sigma}}(k_{v_0})$ and extend it to a unitary character ω' of the locally compact abelian group $\mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A})$. Let ω'_{v_0} be the v_0 -component of ω' . Then $\omega_{\tilde{\sigma}}(\omega'_{v_0})^{-1}$ is a unitary unramified character. Hence $\omega_{\tilde{\sigma}}(\omega'_{v_0})^{-1} = \chi_\nu$ for some $\nu \in \sqrt{-1}\mathfrak{a}_{\mathbf{Z}_{\tilde{\mathbf{L}'}, v_0}, \mathbb{R}}^*$. Since the \mathbf{F} -rational rank of $\mathbf{Z}_{\tilde{\mathbf{L}'}}$ is equal to the F -rational rank of $\mathbf{Z}_{\tilde{\mathbf{L}'}, v_0}$, ν can be seen in $\sqrt{-1}\mathfrak{a}_{\mathbf{Z}_{\tilde{\mathbf{L}'}, \mathbb{R}}}^*$, where $\mathfrak{a}_{\mathbf{Z}_{\tilde{\mathbf{L}'}, \mathbb{R}}}^* := X^*(\mathbf{Z}_{\tilde{\mathbf{L}'}}) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $\omega: \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ defined by $\omega(a) := \omega'(a)q^{H(\nu(a))}$ gives the desired character, where $H: \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}'}}(\mathbb{A}) \rightarrow \mathfrak{a}_{\mathbf{Z}_{\tilde{\mathbf{L}'}, \mathbb{R}}}$ is the global Harish-Chandra homomorphism. \square

As \mathbf{H} is an induced torus, we have

$$\tilde{\mathbf{L}}'(\mathbf{F}) \cong \left(\prod_{i=1}^r (\text{Res}_{\mathbf{E}_i/F} \text{GL}_{n_i})(\mathbf{F}) \right) / \mathbf{H}(\mathbf{F})$$

and similarly for $\tilde{\mathbf{L}}'(\mathbf{F}_v)$ and $\tilde{\mathbf{L}}'(\mathbb{A})$. (In the adelic case, this follows for example from the argument in [Oes84, §III.2.3].) The representation $\tilde{\sigma}$ is a discrete series of an inner form of $\prod_{i=1}^r (\text{Res}_{\mathbf{E}_i/F} \text{GL}_{n_i})(\mathbf{F}_{v_0})$. Let $\tilde{\sigma}'$ be the Jacquet–Langlands transfer of $\tilde{\sigma}$ to the quasisplit form. Since the Jacquet–Langlands correspondence preserves central characters, $\tilde{\sigma}'$ is a representation of $\tilde{\mathbf{L}}'(\mathbf{F}_{v_0})$.

Let V_0 be a finite set of finite places containing v_0 and all (finite) places v such that \mathbf{G}_v is not quasisplit. Let $v_s \notin V_0$ be any finite place.

Lemma 3.48. *There exists a unitary cuspidal automorphic representation $\tilde{\Pi}'$ of $\tilde{\mathbf{L}}'(\mathbb{A})$ such that $\tilde{\Pi}'_{v_0} \cong \tilde{\sigma}'$, $\tilde{\Pi}'_v$ is a discrete series for each $v \in V_0$, and $\tilde{\Pi}'_{v_s}$ is supercuspidal.*

Proof. By Lemma 3.47 there is a unitary character $\omega: \mathbf{Z}_{\tilde{\mathbf{L}}'}(\mathbf{F}) \backslash \mathbf{Z}_{\tilde{\mathbf{L}}'}(\mathbb{A}) \rightarrow \mathbb{C}^\times$ whose v_0 -component is the central character of $\tilde{\sigma}'_{v_0}$. Then the lemma follows by applying the trace formula with central character ω . See the argument in the proof of [Fli87, III.3 Proposition]. \square

Let $\tilde{\Pi}$ be the global Jacquet–Langlands transfer of $\tilde{\Pi}'$ to $\tilde{\mathbf{L}}(\mathbb{A})$ [Bad08, Theorem 5.1]. This is again unitary and exists because $\tilde{\Pi}'_v$ is a discrete series for each $v \in V_0$. (We use again that the Jacquet–Langlands correspondence preserves central characters.) By construction, $\tilde{\Pi}_{v_0} \cong \tilde{\sigma}$, $\tilde{\Pi}_v$ is a discrete series for each $v \in V_0$, $\tilde{\Pi}_v \cong \tilde{\Pi}'_v$ for all $v \notin V_0$, and $\tilde{\Pi}$ is cuspidal, as it is supercuspidal at v_s .

From now on we fix compatible collections of hyperspecial subgroups $K_v \subset \mathbf{G}(\mathbf{F}_v)$ and $K'_v \subset \mathbf{G}'(\mathbf{F}_v)$ for all finite places $v \notin V_0$ (arising from some integral models of \mathbf{G} and \mathbf{G}'). Recall that we have fixed an inner twist $\xi: (\mathbf{G}_{\overline{\mathbf{F}}}, \mathbf{P}_{\overline{\mathbf{F}}}, \mathbf{L}_{\overline{\mathbf{F}}}) \xrightarrow{\sim} (\mathbf{G}'_{\overline{\mathbf{F}}}, \mathbf{P}'_{\overline{\mathbf{F}}}, \mathbf{L}'_{\overline{\mathbf{F}}})$.

Lemma 3.49. *We may increase V_0 such that for all finite places $v \notin V_0$ the isomorphism ξ_v is defined over \mathbf{F}_v , and moreover $\xi_v(K_v) = K'_v$.*

Proof. By spreading out and increasing V_0 we obtain connected reductive $\mathcal{O}_{\mathbf{F}, V_0}$ -group scheme models of $\mathbf{G}, \mathbf{P}, \mathbf{L}$ and $\mathbf{G}', \mathbf{P}', \mathbf{L}'$ (for some finite set of places V_0), which we denote by the same letters. (Here, $\mathcal{O}_{\mathbf{F}, V_0}$ denotes the localization of $\mathcal{O}_{\mathbf{F}}$ away from V_0 .) By increasing V_0 we may assume that $\mathbf{G}(\mathcal{O}_{\mathbf{F}, v}) = K_v$ and $\mathbf{G}'(\mathcal{O}_{\mathbf{F}, v}) = K'_v$ for finite $v \notin V_0$. Increasing V_0 further, the class of ξ in $H^1(\mathbf{F}, \mathbf{L}'/\mathbf{Z}_{\mathbf{G}'})$ spreads out to an étale $\mathbf{L}'/\mathbf{Z}_{\mathbf{G}'}$ -torsor over $\mathcal{O}_{\mathbf{F}, V_0}$, which has to become trivial over $\mathcal{O}_{\mathbf{F}, v}$ for any finite $v \notin V_0$ by Lang’s theorem and the smoothness of $\mathbf{L}'/\mathbf{Z}_{\mathbf{G}'}$. This proves that ξ_v is defined over \mathbf{F}_v . On the other hand, pick a finite extension \mathbf{E}'/\mathbf{F} such that ξ descends to \mathbf{E}' . We increase V_0 such that ξ spreads out to an isomorphism over $\mathcal{O}_{\mathbf{E}', V_0}$ and such that $\mathcal{O}_{\mathbf{F}, V_0} \rightarrow \mathcal{O}_{\mathbf{E}', V_0}$ is étale. Completing at any finite place $v \notin V_0$ we obtain an isomorphism $(\mathbf{G}, \mathbf{P}) \times_{\mathcal{O}_{\mathbf{F}, V_0}} \mathcal{O}_{\mathbf{E}', v} \xrightarrow{\sim} (\mathbf{G}', \mathbf{P}') \times_{\mathcal{O}_{\mathbf{F}, V_0}} \mathcal{O}_{\mathbf{E}', v}$ that arises from an isomorphism over $\mathcal{O}_{\mathbf{F}, v}$ by étale descent (since we already know the cocycle condition holds on the generic fiber). This completes the proof. \square

In particular, for finite places $v \notin V_0$ we may identify $\mathbf{G}(\mathbf{F}_v)$ with $\mathbf{G}'(\mathbf{F}_v)$ (and likewise for $\mathbf{P}, \mathbf{L}, \mathbf{N}$) and K_v with K'_v .

Lemma 3.50. *There exists a (unitary) cuspidal automorphic representation Π of $\mathbf{L}(\mathbb{A})$ which is a quotient of $\tilde{\Pi}|_{\mathbf{L}(\mathbb{A})}$, a (unitary) cuspidal automorphic representation Π' of $\mathbf{L}'(\mathbb{A})$ which is a quotient of $\tilde{\Pi}'|_{\mathbf{L}'(\mathbb{A})}$, and a finite set V_1 of places containing V_0 and the set of infinite places such that*

- $\Pi_{v_0} \cong \sigma$;
- for $v \in V_1 \setminus V_0$, $\Pi_v \cong \Pi'_v$;
- for $v \notin V_1$, $\tilde{\Pi}_v \cong \tilde{\Pi}'_v$ is unramified.

Proof. We fix a maximal compact subgroup $\tilde{K}_v \subset \tilde{\mathbf{L}}(\mathbf{F}_v)$ for infinite v (the subgroup used implicitly in defining automorphic representations on $\tilde{\mathbf{L}}$) and a compact open subgroup $\tilde{K}_v \subset \tilde{\mathbf{L}}(\mathbf{F}_v)$ for finite v such that $\prod_{v \nmid \infty} \tilde{K}_v$ is compact open in

$\tilde{\mathbf{L}}(\mathbb{A}^\infty)$ where \mathbb{A}^∞ is the finite part of \mathbb{A} . Let $\widehat{\tilde{\Pi}}$ be the cuspidal L^2 -automorphic representation of $\tilde{\mathbf{L}}(\mathbb{A})$ whose subspace of $\prod_v \tilde{K}_v$ -finite vectors is $\tilde{\Pi}$. By [LS19, Theorem 1.1] we have an L^2 -automorphic subrepresentation $\widehat{\tilde{\Pi}}_0$ of $\widehat{\tilde{\Pi}}|_{\mathbf{L}(\mathbb{A})}$. For each place v , let $\widehat{\tilde{\Pi}}_v$ (resp. $\widehat{\tilde{\Pi}}_{0,v}$) be the v -component of $\widehat{\tilde{\Pi}}$ (resp. $\widehat{\tilde{\Pi}}_0$). Then $\widehat{\tilde{\Pi}}_v$ is the (unique) unitary completion of $\tilde{\Pi}_v$ and $\widehat{\tilde{\Pi}}_{0,v}$ is a subrepresentation of $\widehat{\tilde{\Pi}}_v|_{\mathbf{L}(\mathbf{F}_v)}$. Likewise we have a compact subgroup $\prod_v \tilde{K}'_v \subset \tilde{\mathbf{L}}'(\mathbb{A})$ and an L^2 -automorphic subrepresentation $\widehat{\tilde{\Pi}}'$ of $\widehat{\tilde{\Pi}}'|_{\mathbf{L}'(\mathbb{A})}$.

Let V_1 be a finite set of places containing V_0 and the set of infinite places such that for any $v \notin V_1$, \mathbf{G}_v is unramified (in particular isomorphic to \mathbf{G}'_v) and $\tilde{\Pi}_v \cong \tilde{\Pi}'_v$ is unramified. Let $\widehat{\sigma}_{v_0}$ be the (unique) unitary completion of σ , so in particular $\widehat{\sigma}_{v_0} \subset \widehat{\tilde{\Pi}}_{v_0}|_{\mathbf{L}(\mathbf{F}_{v_0})}$ (it is the closure of σ). Let $\widehat{\sigma}_v := \widehat{\tilde{\Pi}}'_v$ for $v \in V_1 \setminus V_0$, and note that $\widehat{\tilde{\Pi}}_v \cong \widehat{\tilde{\Pi}}'_v$ for such v . Put $V'_1 := (V_1 \setminus V_0) \cup \{v_0\}$ and for each $v \in V'_1$ define $X_v := \{g_v \in \tilde{\mathbf{L}}(\mathbf{F}_v) \mid \widehat{\tilde{\Pi}}_{0,v} \circ \text{Ad}(g_v) \cong \widehat{\sigma}_v\}$. Note that $\widehat{\tilde{\Pi}}_{0,v}$ and $\widehat{\sigma}_v$ are extended to $\mathbf{L}(\mathbf{F}_v)\mathbf{Z}_{\tilde{\mathbf{L}}}(\mathbf{F}_v)$, as subrepresentations of $\widehat{\tilde{\Pi}}_v|_{\mathbf{L}(\mathbf{F}_v)\mathbf{Z}_{\tilde{\mathbf{L}}}(\mathbf{F}_v)}$, and $\mathbf{L}(\mathbf{F}_v)\mathbf{Z}_{\tilde{\mathbf{L}}}(\mathbf{F}_v) \subset \tilde{\mathbf{L}}(\mathbf{F}_v)$ is open and of finite index. Hence by Clifford theory, X_v is non-empty [LS19, Proposition 4.1.3] and also open as it is $\mathbf{L}(\mathbf{F}_v)\mathbf{Z}_{\tilde{\mathbf{L}}}(\mathbf{F}_v)$ -stable. Since $\tilde{\mathbf{L}}(\mathbf{F}) \subset \prod_{v \in V'_1} \tilde{\mathbf{L}}(\mathbf{F}_v)$ is dense, there exists $g \in \tilde{\mathbf{L}}(\mathbf{F}) \cap \prod_{v \in V'_1} X_v$. Then $\widehat{\Pi} := \widehat{\tilde{\Pi}}_0 \circ \text{Ad}(g)$ is a cuspidal L^2 -automorphic representation, a subrepresentation of $\widehat{\tilde{\Pi}}|_{\mathbf{L}(\mathbb{A})}$ and $\widehat{\Pi}_v \cong \widehat{\sigma}_v$ for any $v \in V'_1$. As $\widehat{\tilde{\Pi}}|_{\mathbf{L}(\mathbb{A})}$ is unitary, $\widehat{\Pi}$ is also a quotient of $\widehat{\tilde{\Pi}}|_{\mathbf{L}(\mathbb{A})}$. Let Π be the subspace of $\prod_v (\tilde{K}_v \cap \mathbf{L}(\mathbf{F}_v))$ -finite vectors in $\widehat{\Pi}$. Then this is an irreducible automorphic representation of $\mathbf{L}(\mathbb{A})$. The surjective map $\widehat{\tilde{\Pi}}|_{\mathbf{L}(\mathbb{A})} \rightarrow \widehat{\Pi}$ is non-zero on $\tilde{\Pi}$ since $\tilde{\Pi} \subset \widehat{\tilde{\Pi}}$ is dense and the image is contained in Π . Therefore we get a non-zero homomorphism $\tilde{\Pi}|_{\mathbf{L}(\mathbb{A})} \rightarrow \Pi$, and it is surjective by irreducibility. Likewise we define Π' and get a surjection $\widehat{\tilde{\Pi}}'|_{\mathbf{L}'(\mathbb{A})} \rightarrow \Pi'$. \square

Take Π and V_1 as in the lemma. Let V be a finite set of places containing V_1 such that for all $v \notin V$ the subgroup $K_v \cap \mathbf{L}(\mathbf{F}_v)$ of $\mathbf{L}(\mathbf{F}_v)$ is hyperspecial and Π_v and Π'_v have nonzero $K_v \cap \mathbf{L}(\mathbf{F}_v)$ -fixed vectors. (Such a set exists by a spreading out argument.) Then we have

- $\Pi_{v_0} \cong \sigma$;
- for $v \in V_0$, Π_v and Π'_v are discrete series;
- for $v \in V_1 \setminus V_0$, $\mathbf{G}_v \cong \mathbf{G}'_v$ and $\Pi_v \cong \Pi'_v$;
- for $v \in V \setminus V_1$, $\mathbf{G}_v \cong \mathbf{G}'_v$ and Π_v, Π'_v are quotients of $\tilde{\Pi}_v|_{\mathbf{L}(\mathbf{F}_v)}$ and $\tilde{\Pi}_v$ is an unramified $\tilde{\mathbf{L}}(\mathbf{F}_v)$ -representation;
- for $v \notin V$, $\mathbf{G}_v \cong \mathbf{G}'_v$ and $\Pi_v \cong \Pi'_v$ have nonzero $K_v \cap \mathbf{L}(\mathbf{F}_v)$ -fixed vectors.

By above, our inner twist ξ gives us natural identifications between $\Phi(\underline{P}, \underline{A}_L)$, $\Phi(\underline{P}', \underline{A}_L')$, $\Phi(\mathbf{P}, \mathbf{A}_L)$, and $\Phi(\mathbf{P}', \mathbf{A}_L')$.

Let $\bar{\mathbf{P}} = \mathbf{L}\bar{\mathbf{N}}$ be the parabolic subgroup of \mathbf{G} opposite to \mathbf{P} . For each $\alpha \in \Phi(\mathbf{P}, \mathbf{A}_L)$ and each place v , we fix Haar measures on $\mathbf{N}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v)$ and $\bar{\mathbf{N}}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v)$ such that

- if $v \notin V_1$, then $\mathbf{N}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v) \cap K_v$ and $\bar{\mathbf{N}}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v) \cap K_v$ have volume 1;
- for the product measure, $(\mathbf{N}(\mathbf{F}) \cap \mathbf{L}_\alpha(\mathbf{F})) \backslash (\mathbf{N}(\mathbb{A}) \cap \mathbf{L}_\alpha(\mathbb{A}))$ and $(\bar{\mathbf{N}}(\mathbf{F}) \cap \mathbf{L}_\alpha(\mathbf{F})) \backslash (\bar{\mathbf{N}}(\mathbb{A}) \cap \mathbf{L}_\alpha(\mathbb{A}))$ have volume 1.

We also choose measures on $\mathbf{N}'(\mathbf{F}_v) \cap \mathbf{L}'_\alpha(\mathbf{F}_v)$ and $\overline{\mathbf{N}}'(\mathbf{F}_v) \cap \mathbf{L}'_\alpha(\mathbf{F}_v)$ which satisfy the analogous conditions. Note that if $v \notin V_1$ then the measure on $\mathbf{N}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v)$ (resp. $\overline{\mathbf{N}}(\mathbf{F}_v) \cap \mathbf{L}_\alpha(\mathbf{F}_v)$) is the same as that on $\mathbf{N}'(\mathbf{F}_v) \cap \mathbf{L}'_\alpha(\mathbf{F}_v)$ (resp. $\overline{\mathbf{N}}'(\mathbf{F}_v) \cap \mathbf{L}'_\alpha(\mathbf{F}_v)$) under our identification of $(\mathbf{G}(\mathbf{F}_v), \mathbf{P}(\mathbf{F}_v), \mathbf{L}(\mathbf{F}_v), \mathbf{N}(\mathbf{F}_v), K_v)$ with $(\mathbf{G}'(\mathbf{F}_v), \mathbf{P}'(\mathbf{F}_v), \mathbf{L}'(\mathbf{F}_v), \mathbf{N}'(\mathbf{F}_v), K'_v)$.

Lemma 3.51. *For $v \in V \setminus V_1$, $j(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) = j(\Pi'_v \delta_{\mathbf{P}'(\mathbf{F}_v)}^s)$ for $s \in \mathbb{C}$.*

Proof. Let $\tilde{\mathbf{T}} \subset \tilde{\mathbf{L}}_v$ be a minimal Levi subgroup (a torus). Take a Borel subgroup $\underline{Q}_{\tilde{\mathbf{L}}_v}$ of $\tilde{\mathbf{L}}_v$ containing $\tilde{\mathbf{T}}$ and a character $\tilde{\chi}$ of $\tilde{\mathbf{T}}$ such that $(\mathrm{n}\text{-Ind}_{\underline{Q}_{\tilde{\mathbf{L}}_v}}^{\tilde{\mathbf{L}}(\mathbf{F}_v)} \tilde{\chi})^{\mathrm{sm}} \rightarrow \tilde{\Pi}_v$. Set $\underline{\mathbf{T}} := \tilde{\mathbf{T}} \cap \mathbf{L}_v$ and let χ be the restriction of $\tilde{\chi}$ to $\underline{\mathbf{T}}$. Let \underline{Q} (resp. \overline{Q}) be the Borel subgroup of \mathbf{G}_v containing \mathbf{N}_v (resp. $\overline{\mathbf{N}}_v$) and such that $\underline{Q} \cap \mathbf{L}_v = \overline{Q} \cap \mathbf{L}_v = \underline{Q}_{\tilde{\mathbf{L}}_v} \cap \mathbf{L}_v$. Then we have $(\mathrm{n}\text{-Ind}_{\underline{Q} \cap \mathbf{L}(\mathbf{F}_v)}^{\mathbf{L}(\mathbf{F}_v)} \chi)^{\mathrm{sm}} = (\mathrm{n}\text{-Ind}_{\underline{Q}_{\tilde{\mathbf{L}}_v}}^{\tilde{\mathbf{L}}(\mathbf{F}_v)} \tilde{\chi})^{\mathrm{sm}}|_{\mathbf{L}(\mathbf{F}_v)} \rightarrow \tilde{\Pi}_v|_{\mathbf{L}(\mathbf{F}_v)} \rightarrow \Pi_v$ and a commutative diagram

$$\begin{array}{ccc} (\mathrm{n}\text{-Ind}_{\underline{Q}}^{\mathbf{G}(\mathbf{F}_v)} \chi \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}} & \xrightarrow{J_{\overline{Q}|\underline{Q}}(\chi \delta_{\mathbf{P}(\mathbf{F}_v)}^s)} & (\mathrm{n}\text{-Ind}_{\overline{Q}}^{\mathbf{G}(\mathbf{F}_v)} \chi \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}} \\ \downarrow & & \downarrow \\ (\mathrm{n}\text{-Ind}_{\mathbf{P}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}} & \xrightarrow{J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)} & (\mathrm{n}\text{-Ind}_{\overline{\mathbf{P}}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}}. \end{array}$$

There is a rational function $c(s)$ such that $J_{\overline{Q}|\underline{Q}}(\chi \delta_{\mathbf{P}(\mathbf{F}_v)}^s) \circ J_{\underline{Q}|\overline{Q}}(\chi \delta_{\mathbf{P}(\mathbf{F}_v)}^s) = c(s)$ and from the above diagram we get $c(s) = j(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)$. The above diagram also holds when we replace Π_v by Π'_v . Hence $c(s) = j(\Pi'_v \delta_{\mathbf{P}'(\mathbf{F}_v)}^s)$ and we get the lemma. \square

Let $s \in \mathbb{C}$ and for $v \notin V$ let $f_{v,\mathrm{ur}} \in (\mathrm{n}\text{-Ind}_{\mathbf{P}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}}$ (resp. $\bar{f}_{v,\mathrm{ur}} \in (\mathrm{n}\text{-Ind}_{\overline{\mathbf{P}}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}}$) be a K_v -fixed vector. We assume $f_{v,\mathrm{ur}}(1) = \bar{f}_{v,\mathrm{ur}}(1) \neq 0$ and for almost all v , this is the vector in Π_v used to define the restricted tensor product $\Pi = \bigotimes'_v \Pi_v$. Then there exists a rational function $c_v(s, \mathbf{P}, \Pi_v)$ such that

$$J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_{v,\mathrm{ur}} = c_v(s, \mathbf{P}, \Pi_v) \bar{f}_{v,\mathrm{ur}}.$$

for each $s \in \mathbb{C}$. An explicit formula of $c_v(s, \mathbf{P}, \Pi_v)$ can be found in [Cas80, 3.1 Theorem]. (Use the diagram in the proof of Lemma 3.51; the product in the cited theorem is then taken over all roots of \mathbf{N}_v .) From the formula, the infinite product $c^V(s, \mathbf{P}, \Pi) = \prod_{v \notin V} c_v(s, \mathbf{P}, \Pi_v)$ is identified with a ratio of products of certain partial L-functions that occur in the Langlands–Shahidi method, cf. [Sha88, (2.7)]. For us, it suffices to know that $c^V(s, \mathbf{P}, \Pi)$ converges if $\mathrm{Re}(s)$ is sufficiently large and gives a meromorphic function on \mathbb{C} . This holds also for $\overline{\mathbf{P}}$. (In this case it converges if $-\mathrm{Re}(s)$ is sufficiently large.)

We have a global intertwining operator $J_{\overline{\mathbf{P}}|\mathbf{P}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s): (\mathrm{n}\text{-Ind}_{\mathbf{P}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} \Pi \delta_{\mathbf{P}(\mathbb{A})}^s)^{\mathrm{sm}} \rightarrow (\mathrm{n}\text{-Ind}_{\overline{\mathbf{P}}(\mathbb{A})}^{\mathbf{G}(\mathbb{A})} \Pi \delta_{\mathbf{P}(\mathbb{A})}^s)^{\mathrm{sm}}$ that converges if $\mathrm{Re}(s)$ is sufficiently large. Put

$$\begin{aligned} f^V &:= \bigotimes_{v \notin V} f_{v,\mathrm{ur}} \in \bigotimes'_{v \notin V} (\mathrm{n}\text{-Ind}_{\mathbf{P}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}}, \\ \bar{f}^V &:= \bigotimes_{v \notin V} \bar{f}_{v,\mathrm{ur}} \in \bigotimes'_{v \notin V} (\mathrm{n}\text{-Ind}_{\overline{\mathbf{P}}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\mathrm{sm}}. \end{aligned}$$

Take $0 \neq f_V := \bigotimes_{v \in V} f_v \in \bigotimes_{v \in V} (\mathfrak{n}\text{-Ind}_{\mathbf{P}(\mathbf{F}_v)}^{\mathbf{G}(\mathbf{F}_v)} \Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s)^{\text{sm}}$. Then if $\text{Re}(s)$ is sufficiently large,

$$\begin{aligned} & J_{\overline{\mathbf{P}}|\mathbf{P}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s)(f^V \otimes f_V) \\ &= \bigotimes_{v \notin V} J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_{v,\text{ur}} \otimes \bigotimes_{v \in V} J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_v \\ &= \bigotimes_{v \notin V} c_v(s, \mathbf{P}, \Pi_v) \overline{f}_{v,\text{ur}} \otimes \bigotimes_{v \in V} J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_v \\ &= c^V(s, \mathbf{P}, \Pi) \overline{f}^V \otimes \bigotimes_{v \in V} J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_v. \end{aligned}$$

Both sides are meromorphic in s , hence this equality holds as meromorphic functions. The same calculation gives

$$\begin{aligned} & J_{\mathbf{P}|\overline{\mathbf{P}}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s) J_{\overline{\mathbf{P}}|\mathbf{P}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s)(f^V \otimes f_V) \\ &= c^V(s, \mathbf{P}, \Pi) c^V(s, \overline{\mathbf{P}}, \Pi) f^V \otimes \bigotimes_{v \in V} J_{\mathbf{P}(\mathbf{F}_v)|\overline{\mathbf{P}}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) J_{\overline{\mathbf{P}}(\mathbf{F}_v)|\mathbf{P}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) f_v \\ &= \left(c^V(s, \mathbf{P}, \Pi) c^V(s, \overline{\mathbf{P}}, \Pi) \prod_{v \in V} j(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) \right) f^V \otimes f_V. \end{aligned}$$

On the other hand, the composition of intertwining operators $J_{\mathbf{P}|\overline{\mathbf{P}}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s) J_{\overline{\mathbf{P}}|\mathbf{P}}(\Pi \delta_{\mathbf{P}(\mathbb{A})}^s)$ is the identity [MW95, Theorem IV.1.10]. Hence we get

$$c^V(s, \mathbf{P}, \Pi) c^V(s, \overline{\mathbf{P}}, \Pi) \prod_{v \in V} j(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) = 1$$

and thus

$$\prod_{v \in V} \mu^{\mathbf{G}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) \sim c^V(s, \mathbf{P}, \Pi) c^V(s, \overline{\mathbf{P}}, \Pi),$$

where \sim denotes equality up to some factor in $\mathbb{R}_{>0}^\times$ that only depends on (\mathbf{G}, \mathbf{L}) (or $(\mathbf{G}', \mathbf{L}')$ below). Notice that if $v \notin V$ then $\mathbf{G}_v \cong \mathbf{G}'_v$ and $\Pi_v \cong \Pi'_v$. Hence, by the same argument for Π' , we have

$$\prod_{v \in V} \mu^{\mathbf{G}'(\mathbf{F}_v)}(\Pi'_v \delta_{\mathbf{P}'(\mathbf{F}_v)}^s) \sim c^V(s, \mathbf{P}, \Pi) c^V(s, \overline{\mathbf{P}}, \Pi).$$

Hence we get

$$\prod_{v \in V} \mu^{\mathbf{G}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) \sim \prod_{v \in V} \mu^{\mathbf{G}'(\mathbf{F}_v)}(\Pi'_v \delta_{\mathbf{P}'(\mathbf{F}_v)}^s).$$

By Lemma 3.51 and $\Pi_v \cong \Pi'_v$ for $v \in V_1 \setminus V_0$, we have

$$\prod_{v \in V_0} \mu^{\mathbf{G}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) \sim \prod_{v \in V_0} \mu^{\mathbf{G}'(\mathbf{F}_v)}(\Pi'_v \delta_{\mathbf{P}'(\mathbf{F}_v)}^s).$$

Since \mathbf{G}'_v is quasisplit and Π'_v is a discrete series for $v \in V_0$, if the right-hand side has a pole at $s \in \mathbb{R}$, then $s \in \mathbb{Q}$ by Proposition 3.41. (Note that the supercuspidal support of Π'_v is generic by Lemma 3.53 below.) On the other hand, if $\mu^{\mathbf{G}(\mathbf{F}_v)}(\Pi_v \delta_{\mathbf{P}(\mathbf{F}_v)}^s) = 0$ for $v \in V_0 \setminus \{v_0\}$ and $s \in \mathbb{R}$, then $s = 0$ by Proposition 3.5. Hence if s is a pole of $\mu^{\mathbf{G}(\mathbf{F}_{v_0})}(\Pi_{v_0} \delta_{\mathbf{P}(\mathbf{F}_{v_0})}^s) = \mu^{\mathbf{G}}(\sigma \delta_{\mathbf{P}}^s)$, then $s \in \mathbb{Q}$. This concludes the proof of Theorem 3.43.

Remark 3.52. With more work it might be possible to show that $\mu^{\mathbf{G}}(\sigma \delta_{\mathbf{P}}^s) = \mu^{\mathbf{G}'}(\sigma' \delta_{\mathbf{P}'}^s)$ for all constituents σ' of $\tilde{\sigma}'$, as in [MS00], [Cho14].

Lemma 3.53. *Any supercuspidal representation of any Levi subgroup of L' is generic.*

Proof. Suppose that τ is a supercuspidal representation of a Levi subgroup \underline{L}'' of \underline{L}' . Then $\underline{L}'' = \tilde{\underline{L}}'' \cap \underline{L}'$ for some Levi subgroup $\tilde{\underline{L}}''$ of $\tilde{\underline{L}}'$ and τ occurs in the restriction of a supercuspidal representation $\tilde{\tau}$ of $\tilde{\underline{L}}''$ [Tad92, §2]. In this way we may assume that $\underline{L}' = \tilde{\underline{L}}'$ (perhaps by varying the nondegenerate character). By a similar (but easier) argument we consider the surjective homomorphism $\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i} \rightarrow \tilde{\underline{L}}'$ to reduce to the case when $\underline{H} = 1$. Finally the claim for the group $\prod_{i=1}^r \text{Res}_{E_i/F} \text{GL}_{n_i}$ follows from [BZ77, 4.4 Theorem]. \square

Proof of Corollary 3.44. As the discrete series σ_1, σ_2 are \tilde{L} -conjugate, there exists a discrete series $\tilde{\sigma}$ of \tilde{L} such that σ_1, σ_2 are irreducible constituents of $\tilde{\sigma}|_L$. We then keep the setup and notation of the proof of Theorem 3.43 above. In particular, $\tilde{\Pi}$ is a cuspidal automorphic representation of \tilde{L} such that $\tilde{\Pi}_{v_0} \cong \tilde{\sigma}$. Then by the proof of Lemma 3.50 we have a cuspidal automorphic representation Π_1 of $\mathbf{L}(\mathbb{A})$ which is a quotient of $\tilde{\Pi}|_{\mathbf{L}(\mathbb{A})}$ such that $\Pi_{1,v_0} \cong \sigma_1$. Hence again by the proof of Lemma 3.50 there exists a cuspidal automorphic representation Π_2 of $\mathbf{L}(\mathbb{A})$ which is a quotient of $\tilde{\Pi}|_{\mathbf{L}(\mathbb{A})}$ such that

- $\Pi_{2,v_0} \cong \sigma_2$;
- $\Pi_{1,v} \cong \Pi_{2,v}$ for any $v \in V_1 \setminus \{v_0\}$;
- for $v \in V \setminus V_1$, $\Pi_{1,v}$ and $\Pi_{2,v}$ are quotients of $\tilde{\Pi}_v|_{\mathbf{L}(\mathbf{F}_v)}$ and $\tilde{\Pi}_v$ is unramified;
- for $v \notin V$, $\mathbf{G}'_v \cong \mathbf{G}'_v$ and $\Pi_v \cong \Pi'_v$ have nonzero $K_v \cap \mathbf{L}(\mathbf{F}_v)$ -fixed vectors.

Then the argument above gives $\mu^G(\sigma_1 \delta_P^s) = \mu^G(\sigma_2 \delta_P^s)$. \square

Remark 3.54. We can extend Corollary 3.44, showing that $J_{P|Q}(\sigma_i \delta_P^s) \circ J_{Q|P}(\sigma_i \delta_P^s)$ is independent of $i \in \{1, 2\}$, where Q is an arbitrary parabolic subgroup with Levi subgroup \underline{L} , not just $Q = \overline{P}$, as in the statement of [Cho14, Working Hypothesis 1.1]. (The above argument still works because of the natural identifications between $\Phi(\underline{P}, \underline{A}_{\underline{L}})$, $\Phi(\underline{P}', \underline{A}_{\underline{L}'})$, $\Phi(\mathbf{P}, \mathbf{A}_{\mathbf{L}})$, and $\Phi(\mathbf{P}', \mathbf{A}_{\mathbf{L}'})$. Alternatively, we can use formulas (12) and (14) in [Wal03, IV.1] to reduce to the case where \underline{P} is a maximal parabolic and Q its opposite.)

3.8. Rationality: rank one groups. Suppose that \underline{G} is simply connected and almost simple of rank one over F . We first recall the classification of such groups, cf. [Tit66, Table II] or [BT84, §4]. First, $\underline{G} \cong \text{Res}_{E/F} \underline{G}'$ for some finite extension E/F and absolutely almost simple group \underline{G}' of rank one, so we may assume that \underline{G} is absolutely almost simple. Let D denote the nonsplit quaternion algebra over F , considered with its canonical involution (fixing precisely F). All (skew-)hermitian forms we consider are non-degenerate.

Proposition 3.55 (Tits). *If \underline{G} is simply connected and absolutely almost simple of rank one over F , then \underline{G} is isomorphic to one of the following:*

- (i) $\text{SL}_2(D')$, where D' is a finite-dimensional central division algebra of dimension d^2 over F ;
- (ii) $\text{SU}(h)$, where E/F is a quadratic extension, h is a hermitian form over E of rank 3 or 4 and Witt index 1 (the groups are quasisplit SU_3 and non-quasisplit SU_4 , respectively);
- (iii) $\text{SU}(h)$, where h is a hermitian form over D of rank 1 or 2 and Witt index 1 (the groups are inner forms of Sp_4 and Sp_6 , respectively);
- (iv) $\widetilde{\text{SU}}(h)$, where h is a skew-hermitian form over D of rank 4 or 5 and Witt index 1 (the groups are an inner form of quasisplit Spin_8 defined by a quadratic extension E/F and an inner form of split Spin_{10} , respectively).

Here, $\widetilde{\text{SU}}(h)$ denotes the simply-connected cover of $\text{SU}(h)$ (in case the Dynkin diagram has type D).

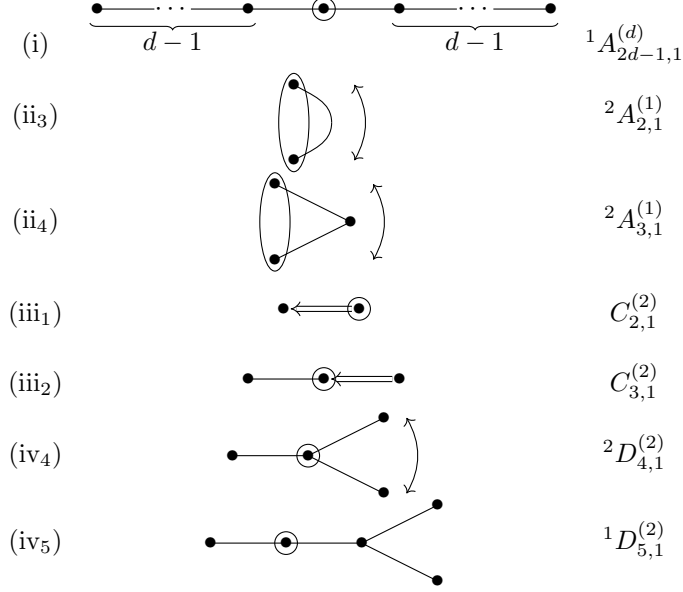


FIGURE 1. Tits indices for the groups in Proposition 3.55, with the case number on the left and the Tits label on the right.

We will write case (ii_r), (iii_r), (iv_r) with $r \in \{1, 2, 3, 4, 5\}$ to refer to the subcase of Proposition 3.55, where the (skew-)hermitian form h has rank r . See Figure 1 for the Tits index in each case. Note that case (ii₄) can alternatively be described by $\widetilde{\text{SU}}(h)$ with h skew-hermitian form over D of rank 2 and Witt index 1, and case (iii₁) can alternatively be described by $\text{Spin}(h)$ with h a quadratic form over F of rank 5 and Witt index 1. Note also that the isomorphism class of \underline{G} is determined by the isomorphism class of $\{D', (D')^{\text{op}}\}$ in case (i), by E/F and r in case (ii_r), by r in case (iii_r), by E/F in case (iv₄), and is unique in case (iv₅).

Let \underline{G}' denote the quasisplit inner form of \underline{G} . Let \underline{L}' be the Levi subgroup of \underline{G}' that corresponds to the minimal Levi \underline{Z} of \underline{G} .

Proposition 3.56. *Keep the above notation. According to the cases of Proposition 3.55 we have:*

- (i) We have $\text{SL}_d \times \text{SL}_d \subset \underline{L}' \subset \text{GL}_d \times \text{GL}_d$.
- (ii₃) We have $\underline{L}' \cong \text{Res}_{E/F} \text{GL}_1$.
- (ii₄) We have $\text{SL}_2 \subset \underline{L}' \subset \text{Res}_{E/F} \text{GL}_1 \times \text{U}_2$.
- (iii₁) We have $\underline{L}' \cong \text{GL}_2$.
- (iii₂) We have $\underline{L}' \cong \text{GL}_2 \times \text{SL}_2$.
- (iv₄) We have $\text{SL}_2 \times \text{Res}_{E/F} \text{SL}_2 \subset \underline{L}' \subset \text{GL}_2 \times \text{Res}_{E/F} \text{GL}_2$.
- (iv₅) We have $\text{SL}_2 \times \text{SL}_4 \subset \underline{L}' \subset \text{GL}_1 \times \text{GL}_2 \times \text{GL}_4$.

Here, U_2 denotes the quasisplit unitary group defined by any (skew-)hermitian form of rank 2 over E .

Proof. This is clear in all but the last two cases.

Case (iv₅): we consider the split group GSpin_{10} whose derived subgroup is \underline{G}' and show that the corresponding Levi is contained between $\text{SL}_2 \times \text{SL}_4$ and $\text{GL}_1 \times \text{GL}_2 \times \text{GL}_4$, which implies the claim by intersecting with the derived subgroup. The Levi subgroup in question is $\text{GL}_2 \times \text{GSpin}_6$ [Asg02, Theorem 2.7] and

$\mathrm{GSpin}_6 \cong \{(g_1, g_2) \in \mathrm{GL}_1 \times \mathrm{GL}_4 : g_1^2 = \det(g_2)\}$ by [AC17, Proposition 2.1], so we are done.

Case (iv₄): as in the previous case we consider instead the quasisplit group GSpin_8^* defined by the quadratic extension E/F . (We note that the group of automorphisms of the root datum of GSpin_{2n} is $\mathbb{Z}/2 \times \mathbb{Z}/2$, unlike some claims in the literature, but we use the unique non-trivial automorphism that is trivial on the center.) Explicitly, it interchanges e_0 with $e_0 + e_n$, e_n with $-e_n$, and fixes e_1, \dots, e_{n-1} (in the description of [Asg02, Proposition 2.4]). It follows immediately from the description of the root datum that the Levi subgroup in question is $\mathrm{GL}_2 \times \mathrm{GSpin}_4^*$. We conclude by Lemma 3.58 below. \square

Lemma 3.57. *We have*

$$\mathrm{U}_2 \cong \left(\frac{\mathrm{GL}_2 \times \mathrm{Res}_{E/F} \mathrm{GL}_1}{\mathrm{GL}_1} \right)^{N=1},$$

where GL_1 is embedded antidiagonally and N is the product of the determinant on GL_2 and the norm map $\mathrm{Res}_{E/F} \mathrm{GL}_1 \rightarrow \mathrm{GL}_1$.

Proof. It suffices to show that

$$(3.7) \quad \mathrm{GU}_2 \cong \frac{\mathrm{GL}_2 \times \mathrm{Res}_{E/F} \mathrm{GL}_1}{\mathrm{GL}_1}$$

such that the multiplier on the left corresponds to N on the right. It is convenient to work with the skew-hermitian form defined by the matrix $J := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. Then over E the group GU_2 becomes identified with $\mathrm{GL}_2 \times \mathrm{GL}_1$, with the nontrivial element $\tau \in \mathrm{Gal}(E/F)$ acting by the automorphism $(g, \lambda) \mapsto (\lambda \cdot J^{-1} \cdot {}^t g^{-1} \cdot J, \lambda)$. Similarly the right-hand side of (3.7) becomes identified with $\frac{\mathrm{GL}_2 \times \mathrm{GL}_1 \times \mathrm{GL}_1}{\mathrm{GL}_1}$ with GL_1 embedded via $x \mapsto \overline{(x^{-1}, x, x)}$ and τ acting by $\overline{(g, x, y)} \mapsto \overline{(g, y, x)}$. It is then easily verified that the maps $(g, \lambda) \mapsto \overline{(g, 1, \lambda(\det g)^{-1})}$ and $\overline{(g, x, y)} \mapsto \overline{(gx, (\det g)xy)}$ are inverse isomorphisms that commute with the action of τ . \square

Lemma 3.58. *The quasisplit group GSpin_4^* defined by the quadratic extension E/F is isomorphic to $\{g \in \mathrm{Res}_{E/F} \mathrm{GL}_2 : \det(g) \in \mathrm{GL}_1 \subset \mathrm{Res}_{E/F} \mathrm{GL}_1\}$.*

Proof. We consider a variant of the argument in [AC17, Proposition 2.1]. We work over an algebraic closure of F that contains E and identify GSpin_4 with $\underline{H} := \{(g_1, g_2) \in \mathrm{GL}_2^2 : \det(g_1) = \det(g_2)\}$ by identifying the root data as follows. Consider the maximal torus $\underline{T} := \{t = \begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} c & \\ & d \end{pmatrix}\} \in \mathrm{GL}_2^2 : ab = cd\}$ of \underline{H} and define the basis $e_i \in X^*(\underline{T})$ ($0 \leq i \leq 2$) by $e_0(t) := d$, $e_1(t) := ad^{-1} = cb^{-1}$, $e_2(t) := bd^{-1} = ca^{-1}$. It is straightforward to verify that this identifies the root datum of $(\underline{H}, \underline{T})$ with that of GSpin_4 as given in [Asg02, Proposition 2.4] and that it identifies the action of $\mathrm{Gal}(E/F)$ on the former root datum (interchanging (a, b) with (c, d)) with the action on the latter (exchanging e_0 with $e_0 + e_2$, e_2 with $-e_2$ and fixing e_1). This completes the proof. \square

Corollary 3.59. *Suppose that the adjoint group $\underline{G}^{\mathrm{ad}}$ is almost simple of rank one over F . Let σ be a unitary supercuspidal smooth representation of Z over \mathbb{C} . If $\mu^G(\sigma \delta_B^s)$ has a pole at $s = s_0 \in \mathbb{R}$, then $s_0 \in \mathbb{Q}$.*

Proof. Let $\underline{G}^{\mathrm{sc}}$ be the simply-connected cover of $\underline{G}^{\mathrm{der}}$. By Proposition 3.6 (and replacing σ by an irreducible constituent of $\sigma|_{\underline{G}^{\mathrm{sc}}}$) we may assume that \underline{G} is semisimple and simply connected. As mentioned above, by replacing F by a finite extension we may assume that \underline{G} is moreover absolutely almost simple. Then the result follows by combining Theorem 3.43 with Propositions 3.55 and 3.56. \square

Remark 3.60. This result was previously known in case (i) by [Tad90] (cf. §3.4), in case (ii₃) by [Key84, §7], in case (ii₄) by [Kon03, §3.1], and in case (iii₁) by [MS00, §3]. As far as we know, it is new in cases (iii₂), (iv₄), and (iv₅).

Remark 3.61. Suppose that $\underline{G}^{\text{ad}}$ is almost simple of rank one over F . In the setting of Corollary 3.59 we can moreover bound the denominator of s_0 , and hence narrow down the set of possible s_0 to an explicit finite set by Proposition 3.5(ii), as follows:

- in case (iii₂) and (iv₄), $s_0 \in \frac{1}{20}\mathbb{Z}$;
- in case (iv₅), $s_0 \in \frac{1}{28}\mathbb{Z}$.

(We recall that $s_0 \in \{\pm \frac{1}{2e} \mid e \mid d\}$ in case (i) [Tad90], $s_0 \in \frac{1}{4}\mathbb{Z}$ in case (ii₃) [Key84], $s_0 \in \frac{1}{6}\mathbb{Z}$ in case (ii₄) [Kon03], and $s_0 \in \{0, \pm \frac{1}{6}, \pm \frac{1}{2}\}$ in case (iii₁) [MS00].) To see this, we may again reduce to the case where \underline{G} is simply connected and absolutely almost simple. By the proof of Theorem 3.43 it suffices to consider the quasisplit inner form \underline{G}' with parabolic subgroup $\underline{P}' = \underline{L}'\underline{N}'$, where \underline{L}' is described in Proposition 3.56. By Remark 3.42 it suffices to show that $\langle 2\rho_{\underline{P}'}, \gamma^\vee \rangle_{\text{abs}}$ is an integer dividing 10 (resp. 14) for γ a root of $\underline{A}_{\underline{Z}'}$ in \underline{N}' . This is an explicit computation. (In fact, with more work we can show that $s_0 \in \frac{1}{10}\mathbb{Z}$ in cases (iii₂) and (iv₄) and $s_0 \in \frac{1}{14}\mathbb{Z}$ in case (iv₅) using Remark 3.40.)

3.9. Proof of Theorem 3.33. Now we prove Theorem 3.33. We always assume Assumption 2.14 in this subsection.

Suppose \underline{G} is an arbitrary connected reductive group, $\underline{B} = \underline{Z}\underline{U}$ a choice of minimal parabolic subgroup, \underline{S} the maximal split subtorus of \underline{Z} . We first reduce to the case where $F = \mathbb{Q}_p$, so we can apply the results of section 2.8.

Lemma 3.62. *Theorem 3.33 holds over arbitrary F if and only if it holds over $F = \mathbb{Q}_p$.*

Proof. It suffices to show that Theorem 3.33 holds for (G, B, σ) if and only if it holds for $(\text{Res } G, \text{Res } B, \sigma)$, where $\text{Res} := \text{Res}_{F/\mathbb{Q}_p}$ denotes Weil restriction of scalars. First note that $\text{Res } \underline{B}$ is a minimal parabolic subgroup of $\text{Res } \underline{G}$ with Levi subgroup $\text{Res } \underline{Z}$. (This can be verified by extending scalars to an algebraic closure.) The split torus \underline{S} is obtained by base change from a unique split torus \underline{S}_0 over \mathbb{Q}_p , hence the adjunction gives a homomorphism $\underline{S}_0 \rightarrow \text{Res } \underline{G}$, which identifies \underline{S}_0 with a maximal split torus of $\text{Res } \underline{G}$ (see, e.g. [HR20, §4.2]) that is moreover contained in the minimal Levi subgroup $\text{Res } \underline{Z}$. Noting that $\text{Lie}(\text{Res } \underline{G})$ equals $\text{Lie } \underline{G}$ considered as Lie algebra over \mathbb{Q}_p , we see that $\Phi(\underline{G}, \underline{S})$ is identified with $\Phi(\text{Res } \underline{G}, \underline{S}_0)$ via the map sending $\mu : \underline{S} \rightarrow \mathbb{G}_m$ to the restriction of $\text{Res } \mu$ to \underline{S}_0 (noting that $\text{Res } \mu$ factors through the maximal split torus $\mathbb{G}_m \hookrightarrow \text{Res } \mathbb{G}_m$); equivalently, roots are identified in $X^*(\underline{S}_0) \cong X^*(\underline{S})$ (under extension of scalars). The same is true for positive and simple roots. By comparing roots, we see that we have an equality of Levi subgroups $\text{Res } \underline{L}_\alpha = \underline{L}_{\text{Res } \alpha|_{\underline{S}_0}}$ for any $\alpha \in \Phi(\underline{G}, \underline{S})$. Therefore, the condition that $\sigma|_{\underline{Z} \cap \underline{L}'_\alpha}$ is trivial for some simple root σ is the same for both G and $\text{Res } G$. On the other hand, using that the coroot α^\vee is uniquely determined by having image in \underline{L}'_α and satisfying $\langle \alpha, \alpha^\vee \rangle = 2$, we deduce that coroots are identified in $X_*(\underline{S}_0) \cong X_*(\underline{S})$. We claim that $e(\chi|_{\underline{S}}) = [F : \mathbb{Q}_p]e(\chi)$ in $X^*(\underline{S}) \cong X^*(\underline{S}_0)_\mathbb{R}$ for any $\chi : \underline{S} \rightarrow C^\times$ (and likewise for characters of A_L). It suffices to check this when $|\chi|_C = |\nu|_F$ for some $\nu \in X^*(\underline{S})$, in which case it follows from $|\cdot|_F = |\cdot|_{\mathbb{Q}_p}^{[F:\mathbb{Q}_p]}$ on \mathbb{Q}_p^\times . Therefore, $e(\chi|_{\underline{S}})$ is dominant if and only if $e(\chi|_{\underline{S}_0})$ is dominant. \square

From now on we assume that $F = \mathbb{Q}_p$.

Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi_{\text{red}}^+} n_\alpha \alpha \in X^*(\underline{S})$, where $n_\alpha := \dim U_{(\alpha)} + \dim U_{(2\alpha)} \in \mathbb{Z}_{>0}$. Note that n_α only depends on the Weyl group orbit of α .

Lemma 3.63. *For any $\alpha \in \Phi_{\text{red}}^+$ we have $\langle \rho, \alpha^\vee \rangle \geq n_\alpha$ with equality if and only if α is simple.*

Proof. If α is simple, then $s_\alpha(\Phi_{\text{red}}^+) = (\Phi_{\text{red}}^+ \setminus \{\alpha\}) \cup \{-\alpha\}$ and the proof follows from

$$-\langle \rho, \alpha^\vee \rangle = \langle \rho, s_\alpha(\alpha^\vee) \rangle = \langle s_\alpha(\rho), \alpha^\vee \rangle = \frac{1}{2} \sum_{\beta \in s_\alpha(\Phi_{\text{red}}^+)} n_\beta \langle \beta, \alpha^\vee \rangle = \langle \rho, \alpha^\vee \rangle - 2n_\alpha.$$

For general α , we will work in the reduced root system Φ_{red} , so we can write $\alpha^\vee = \sum_{i=1}^r \alpha_i^\vee$ with α_i simple and therefore $\langle \rho, \alpha^\vee \rangle = \sum_{i=1}^r n_{\alpha_i}$ by the previous paragraph. To show $\langle \rho, \alpha^\vee \rangle \geq n_\alpha$ it thus suffices to show that at least one α_i is in the same Weyl group orbit as α , as this implies that $n_{\alpha_i} = n_\alpha$. Note also that once we have shown this, equality can only hold if $r = 1$, i.e. α simple.

By dualizing it suffices to show the following. Suppose that Φ is a reduced root system. Then for any $\alpha \in \Phi^+$ with $\alpha = \sum_{\gamma \in \Delta} c_\gamma \gamma$ there is a $\gamma \in \Delta$ with $c_\gamma > 0$ and γ is in the same Weyl group orbit as α . Let $I := \{\gamma \in \Delta : c_\gamma \neq 0\}$ and let $\Phi_I := \mathbb{Z}I \cap \Phi$ denote the sub-root system generated by I . Since $\alpha \in \Phi_I$ lies in the Weyl group orbit of a simple root γ of Φ_I^+ (i.e. $\gamma \in I$), we are done. \square

Recall that \underline{T}' denotes a maximal split torus of (the split group) \underline{G}_C containing \underline{S}_C and recall that $\tilde{\Phi}$ denotes the roots of $(\underline{G}_C, \underline{T}')$. Let $\tilde{\Delta} \subset \tilde{\Phi}$ denote a choice of simple roots so that $\Delta \subset \tilde{\Delta}|_{\underline{S}} \subset \Delta \cup \{0\}$. Let $\tilde{\Delta}_0 := \{\tilde{\gamma} \in \tilde{\Delta} : \tilde{\gamma}|_{\underline{S}} = 1\}$. For $\alpha \in \Delta$ let $\mathcal{O}(\alpha) := \{\tilde{\gamma} \in \tilde{\Delta} : \tilde{\gamma}|_{\underline{S}} = \alpha\}$.

Lemma 3.64. *Suppose $\alpha \in \Delta$ and write $\alpha^\vee = \sum_{\tilde{\gamma} \in \tilde{\Delta}} c_{\tilde{\gamma}} \tilde{\gamma}^\vee$ with $c_{\tilde{\gamma}} \in \mathbb{Z}$. Then $c_{\tilde{\gamma}} \geq 0$ and $c_{\tilde{\gamma}} \neq 0$ if and only if $\tilde{\gamma}$ lies in the smallest union of connected components of $\tilde{\Delta}_0 \sqcup \mathcal{O}(\alpha)$ that contain $\mathcal{O}(\alpha)$.*

Note that α^\vee can be expressed in the above form because we can pass to the simply-connected cover of the derived subgroup of \underline{G} (in which case $X_*(\underline{T}') = \mathbb{Z}\tilde{\Delta}^\vee$).

Proof. By passing to the Levi subgroup L_α we may assume that \underline{G} has semisimple rank 1 (i.e. $\tilde{\Delta} = \tilde{\Delta}_0 \sqcup \mathcal{O}(\alpha)$). By comparing with the simply-connected cover of the derived subgroup we may assume that \underline{G} is simply connected. Factoring $\underline{G} = \underline{G}^{\text{is}} \times \underline{G}^{\text{an}}$ with $\underline{G}^{\text{is}}$ almost simple and isotropic and $\underline{G}^{\text{an}}$ anisotropic, we may assume that \underline{G} is almost simple, and we need to show that $c_{\tilde{\gamma}} \neq 0$ for all $\tilde{\gamma} \in \tilde{\Delta}$.

Note that the $c_{\tilde{\gamma}}$ are determined by the formulas

$$\langle \tilde{\beta}, \alpha^\vee \rangle = \sum_{\tilde{\gamma} \in \tilde{\Delta}} c_{\tilde{\gamma}} \langle \tilde{\beta}, \tilde{\gamma}^\vee \rangle = \begin{cases} 2 & \text{if } \tilde{\beta} \in \mathcal{O}(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\alpha^\vee = 2 \sum_{\tilde{\beta} \in \mathcal{O}(\alpha)} \varpi_{\tilde{\beta}}^\vee$, where $\varpi_{\tilde{\beta}}^\vee$ denotes the fundamental coweight corresponding to $\tilde{\beta}$.

Dually, it suffices to show that if we express a fundamental weight $\varpi_{\tilde{\beta}} = \sum_{\tilde{\gamma} \in \tilde{\Delta}} d_{\tilde{\gamma}} \tilde{\gamma}$ in terms of simple roots, then $d_{\tilde{\gamma}} \geq 0$ for all $\tilde{\gamma}$ and $d_{\tilde{\gamma}} > 0$ if $\tilde{\gamma}$ lies in the connected component of $\tilde{\Delta}$ that contains $\tilde{\beta}$. The first claim is true since fundamental weights form acute or right angles with each other. If the second claim is false, we can pick (adjacent) $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \tilde{\Delta}$ such that $d_{\tilde{\gamma}_1} = 0$, $d_{\tilde{\gamma}_2} > 0$ and $\langle \tilde{\gamma}_2, \tilde{\gamma}_1^\vee \rangle < 0$. Then $\langle \varpi_{\tilde{\beta}}, \tilde{\gamma}_1^\vee \rangle = \sum_{\tilde{\gamma} \in \tilde{\Delta}} d_{\tilde{\gamma}} \langle \tilde{\gamma}, \tilde{\gamma}_1^\vee \rangle < 0$, contradiction. \square

Recall that L'_α is the group generated by $U \cap L_\alpha$ and $\bar{U} \cap L_\alpha$.

Lemma 3.65. *Suppose that V is an irreducible algebraic representation of G over C , so that V^U is an irreducible algebraic representation of Z . Let $\omega_{V^U} : S \rightarrow C^\times$*

denote the central character of V^U and let $\underline{P} = \underline{L}N$ be a standard parabolic subgroup. Then the following are equivalent:

- (i) V^N is trivial on L' ;
- (ii) V^U is trivial on $Z \cap L'$;
- (iii) $\omega_{V^U} \circ \alpha^\vee = 1$ for all $\alpha \in \Delta_L$.

Here, Δ_L denotes the simple roots of \underline{L} .

Proof. It is clear that (i) \Rightarrow (ii) \Rightarrow (iii). To show (iii) \Rightarrow (i), by taking (G, V) to be (L, V^N) , we may assume that $\underline{L} = \underline{G}$. As in [AHHV17, II.4] let $\underline{G}^{\text{sc}}$ denote the simply-connected cover of the derived subgroup of \underline{G} , which is a direct product of almost simple groups, precisely one of which is isotropic (of rank 1). Let $\underline{G}^{\text{is}}$ denote the unique isotropic almost simple factor of $\underline{G}^{\text{sc}}$. Then G' is the image of $G^{\text{is}} = (G^{\text{is}})'$ under the natural morphism $\iota: \underline{G}^{\text{is}} \rightarrow \underline{G}$ (by Kneser–Tits, cf. [AHHV17, II.4]). As ι is an isomorphism $\underline{U}^{\text{is}} \rightarrow \underline{U}$ and $\iota^{-1}(Z) = Z^{\text{is}}$ is a minimal Levi subgroup, we reduce to the case where \underline{G} is a product of isotropic simply-connected groups.

Let $\mu \in X^*(\underline{T}')$ denote the highest weight of V (or V^U), so that ω_{V^U} is given by $\mu|_S$. In particular, $\langle \mu, \tilde{\gamma}^\vee \rangle \geq 0$ for all $\tilde{\gamma} \in \tilde{\Delta}$. By our assumption on \underline{G} no connected component of $\tilde{\Delta}$ is contained in $\tilde{\Delta}_0$, so for each $\tilde{\gamma} \in \tilde{\Delta}_0$ there exists $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}_0$ such that $\tilde{\gamma}, \tilde{\alpha}$ lie in the same connected component of $\tilde{\Delta}_0 \sqcup \mathcal{O}(\tilde{\alpha}|_S)$. By Lemma 3.64 we deduce that $\omega_{V^U} \circ \alpha^\vee = 1$ for all $\alpha \in \Delta$ if and only if $\langle \mu, \tilde{\gamma}^\vee \rangle = 0$ for all $\tilde{\gamma} \in \tilde{\Delta}$ if and only if $\mu = 0$ if and only if V is trivial on $G = G'$. \square

Remark 3.66. Note that in the setting of Lemma 3.65 we have $\omega_{V^U} \circ \alpha^\vee = (\cdot)^{m_\alpha}$ for some $m_\alpha \in \mathbb{Z}_{\geq 0}$ for any $\alpha \in \Delta$.

Proof of Theorem 3.33. We first prove that the conditions in (ii) and (iii) are necessary. Assume that $\underline{L}(\sigma') \in \mathcal{O}^{P_1}$ for a parabolic subgroup $\underline{P}_1 \supseteq \underline{P}$ with Levi subgroup $\underline{L}_1 \supset \underline{Z}$. Then $\sigma_1 := \underline{L}_{L_1}(\sigma')$ is finite-dimensional by Corollary 2.12 and we have $(\sigma_1)_{N \cap L_1} \cong \sigma$. By Frobenius reciprocity we have $\sigma_1 \hookrightarrow (\text{Ind}_{P \cap L_1}^{L_1} \sigma)^{\text{cts}}$, which is closed as σ_1 is finite-dimensional. Then by applying the functor $(\text{Ind}_{P_1}^G \cdot)^{\text{cts}}$, we get $(\text{Ind}_{P_1}^G \sigma_1)^{\text{cts}} \hookrightarrow (\text{Ind}_P^G \sigma)^{\text{cts}}$, so $(\text{Ind}_P^G \sigma)^{\text{cts}}$ is reducible. The proof of the “if” part of (iii) is similar (see the introduction).

In the remaining proof we may freely replace C by a finite extension because this preserves the reducibility, and we will do so without further comment.

Recall that we may assume that $F = \mathbb{Q}_p$. Recall that we have defined $e(\chi) = e_S(\chi) \in \mathfrak{a}_{\mathbb{Z}, \mathbb{R}}^*$ for a character $\chi: S \rightarrow C^\times$, and also that if $\chi \in X^*(\underline{S})$ is algebraic, thought of as $\chi: S \rightarrow \mathbb{Q}_p^\times \subset C^\times$, then $e(\chi) = \chi$. It follows that $e(|\chi|_{\mathbb{Q}_p}) = -\chi$ for $\chi \in X^*(\underline{S})$. In this proof we write $e(\cdot)$ for $e(\cdot|_S)$.

We may assume that $\underline{G}^{\text{der}}$ is simply connected, by taking a z -extension. By Lemma 2.51 we can write $\sigma \cong \sigma_0 \otimes \tau$ with $\sigma_0 \in \mathcal{O}^L$ and τ a smooth representation of L such that $\underline{L}(\sigma'_0) \in \mathcal{O}^P$ is equimaximal. Let $\underline{Q} = \underline{L}_Q \underline{N}_Q$ denote the maximal parabolic subgroup for $\underline{L}(\sigma'_0)$. By Corollary 2.54 we deduce that $(\text{Ind}_{P \cap L_Q}^{L_Q} \tau)^{\text{sm}}$ is reducible. We may relabel L_Q as G and assume without loss of generality that $Q = G$. In particular, by Lemma 2.22 $\underline{L}(\sigma'_0) \in \mathcal{O}^G$ is algebraic up to twist by a locally analytic character $\psi: G \rightarrow C^\times$. Twisting σ by ψ and using Lemma 2.4, without loss of generality, $\underline{L}(\sigma'_0) \in \mathcal{O}^G$ is algebraic. Then $(\sigma'_0)^{U \cap L} \cong \underline{L}(\sigma'_0)^U$ is an irreducible algebraic representation of Z . Let $\omega_0: S \rightarrow C^\times$ denote its central character, which is algebraic (factoring through $\mathbb{Q}_p^\times \subset C^\times$) and dominant by Remark 3.66. So if we write $\omega_0 = -\lambda \in X^*(\underline{S})$, then $\langle \lambda, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Delta$.

Note that τ is trivial on L' by Lemma 2.4. Let $\xi := \tau|_Z$, which is absolutely irreducible as $ZL' = L$.

Taking coinvariants, we see that $\sigma_{U \cap L} \cong (\sigma_0)_{U \cap L} \otimes \xi$. As $(\sigma_0)_{U \cap L}$ is \mathfrak{z}_C -simple and ξ is absolutely irreducible smooth Z -module, we see that $\sigma_{U \cap L}$ is absolutely irreducible by Proposition 2.24. Let $\omega_\xi, \omega_{\sigma_{U \cap L}} : S \rightarrow C^\times$ denote the central characters of $\xi, \omega_{U \cap L}$, respectively. By applying $e(\cdot)$ to central characters we obtain that

$$(3.8) \quad \lambda + e(\omega_\xi) = e(\omega_{\sigma_{U \cap L}}).$$

For $\beta \in \Delta_L$ we have $\langle e(\omega_\xi), \beta^\vee \rangle = e_{\mathbb{G}_m}(\omega_\xi \circ \beta^\vee) = 0$ (as ξ is trivial on $Z \cap L'$).

We claim that $e(\omega_{\sigma_{U \cap L}}) - \lambda$ is dominant. Assume that $e(\omega_{\sigma|_{A_L}})$ is dominant (which is implied by $e(\omega_{\sigma_{U \cap L}})$ being dominant, as remarked above). For $\alpha \in \Delta_L$, as $\langle e(\omega_\xi), \alpha^\vee \rangle = 0$, we have $\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, \alpha^\vee \rangle = 0$. On the other hand, let $\alpha \in \Delta \setminus \Delta_L$ and take $a \in \mathfrak{a}_{L, \mathbb{R}}, b \in \mathfrak{a}_{Z, \mathbb{R}}^L = \mathbb{R}\Delta_L^\vee$ such that $\alpha^\vee = a + b$. By Lemma 3.2(i), we have $b \in \mathbb{R}_{\leq 0}\Delta_L^\vee$. Therefore $\langle \lambda, a \rangle = \langle \lambda, \alpha^\vee - b \rangle \leq 0$ since λ is antidominant. Since $\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, b \rangle = \langle e(\omega_\xi), b \rangle = 0$ and $\langle e(\omega_{\sigma_{U \cap L}}), a \rangle = \langle e(\omega_{\sigma|_{A_L}}), a \rangle = \langle e(\omega_{\sigma|_{A_L}}), \alpha^\vee \rangle \geq 0$, we have $\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, \alpha^\vee \rangle = \langle e(\omega_{\sigma_{U \cap L}}), a \rangle - \langle \lambda, a \rangle \geq 0$. Therefore $e(\omega_{\sigma_{U \cap L}}) - \lambda$ is dominant.

We embed $(\text{Ind}_P^G \tau)^{\text{sm}}$ as subrepresentation of $(\text{Ind}_B^G \xi)^{\text{sm}} = (\text{n-Ind}_B^G \xi \delta_B^{-1/2})^{\text{sm}}$. We can write $\delta_B(x) = |(2\rho)(x)|_{\mathbb{Q}_p}$ for $x \in S$ and so $\delta_{B \cap L_\alpha}(x) = |\alpha(x)|_{\mathbb{Q}_p}^{n_\alpha}$ for $\alpha \in \Phi_{\text{red}}^+$. (Here $\rho \in X^*(S)$ and $n_\alpha \in \mathbb{Z}_{>0}$ are as in Lemma 3.63.)

Then

$$(3.9) \quad e(\omega_\xi \delta_B^{-1/2}) = (e(\omega_{\sigma_{U \cap L}}) - \lambda) + \rho,$$

as $e(\delta_B) = -2\rho$.

If $w \in N_G(Z)$ satisfies $\xi \delta_B^{-1/2} \cong w(\xi \delta_B^{-1/2})$, then by (3.9) we deduce that $(e(\omega_{\sigma_{U \cap L}}) - \lambda) + \rho$ is fixed by w (here we regard w as an element of the Weyl group), which implies $w \in Z$, as $e(\omega_{\sigma_{U \cap L}}) - \lambda$ is dominant and ρ is strictly dominant. Hence $\xi \delta_B^{-1/2}$ is G -regular, so μ^G has a pole at $\xi \delta_B^{-1/2}$ by Proposition 3.8, and therefore μ^{L_α} has a pole at $\xi \delta_B^{-1/2}$ for some $\alpha \in \Phi_{\text{red}}^+$ by the product formula (Proposition 3.4).

We will now work over $\overline{C} \cong \mathbb{C}$ (fixing a field isomorphism arbitrarily) and extend the absolute value $|\cdot|_C$ uniquely to \overline{C} . Write $\xi \delta_B^{-1/2} \cong \xi_u \chi$ with ξ_u unitary over \mathbb{C} and χ a positive real unramified character of Z . We can write $\chi = \delta_{B \cap L_\alpha}^s \eta$ with $s \in \mathbb{R}$ and η a positive real unramified character of L_α (as $\mathfrak{a}_{Z, \mathbb{R}}^* = \mathfrak{a}_{L_\alpha, \mathbb{R}}^* \oplus (\mathfrak{a}_{Z, \mathbb{R}}^{L_\alpha})^*$ and α spans $(\mathfrak{a}_{Z, \mathbb{R}}^{L_\alpha})^*$). By Proposition 3.5 we have $\xi_u \cong s_\alpha(\xi_u)$ where $s_\alpha \in N_{L_\alpha}(Z)/Z$ is the non-trivial element. Thus $(\omega_{\xi_u} \circ \alpha^\vee)^2 = 1$, where $\omega_{\xi_u} : S \rightarrow \mathbb{C}^\times$ is the central character of ξ_u . In particular, $\langle e(\omega_{\xi_u}), \alpha^\vee \rangle = e_{\mathbb{G}_m}(\omega_{\xi_u} \circ \alpha^\vee) = 0$. By Proposition 3.5 we know that $-1/2 \leq s \leq 1/2$. By Corollary 3.59 (applied to L_α) we know that $s \in \mathbb{Q}$, so $\delta_{B \cap L_\alpha}^s(x) = |\alpha(x)|_{\mathbb{Q}_p}^{sn_\alpha}$ (taking values in $p^\mathbb{Q} \subset \mathbb{R}_{>0}^\times$). Hence $e(\delta_{B \cap L_\alpha}^s) = -sn_\alpha \alpha$. By applying $e(\cdot)$ to the central character of $\xi \delta_B^{-1/2} \cong \xi_u \chi = \xi_u \delta_{B \cap L_\alpha}^s \eta$, pairing with $-\alpha^\vee$, and using (3.9) we deduce

$$\langle \lambda - \rho - e(\omega_{\sigma_{U \cap L}}), \alpha^\vee \rangle = 2sn_\alpha.$$

As $e(\omega_{\sigma_{U \cap L}}) - \lambda$ is dominant, and by Lemma 3.63 we get

$$0 \geq -\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, \alpha^\vee \rangle = \langle \rho, \alpha^\vee \rangle + 2sn_\alpha \geq n_\alpha(2s + 1).$$

Since $s \geq -1/2$, we get $\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, \alpha^\vee \rangle = 0$, $\langle \rho, \alpha^\vee \rangle = n_\alpha$ and $s = -1/2$. By Lemmas 3.63 and Proposition 3.7 we deduce that α is simple and ξ_u is trivial on $Z \cap L'_\alpha$. Since $\delta_B = \delta_{B \cap L_\alpha}(\delta_{P_\alpha}|_Z)$ we get that $\xi \cong \xi_u \eta(\delta_{P_\alpha}|_Z)^{1/2}$ is trivial on $Z \cap L'_\alpha$.

Case 1: $\alpha \notin \Delta_L$. Let \underline{L}_1 be the smallest Levi subgroup containing \underline{L} and \underline{L}_α , and let $\underline{L}_1 \underline{N}_1$ be the standard parabolic subgroup with Levi subgroup \underline{L}_1 . Then since $\xi = \tau|_Z$ is trivial on $Z \cap L'_\alpha$, τ has an extension τ_1 to L_1 [AHHV17, II.7 Proposition]

with $N \cap L_1$ acting trivially. Then $\underline{L}(\sigma') \cong \underline{L}(\sigma'_0) \otimes \tau'$ in \mathcal{O}^P by Lemma 2.13. By Lemma 2.8 and as $\underline{L}(\sigma'_0) \in \mathcal{O}^G$ we have $\underline{L}(\sigma') \cong \underline{L}(\sigma'_0) \otimes \tau'_1$ lies in \mathcal{O}^{P_1} .

Assume moreover $e(\omega_{\sigma_{U \cap L}})$ is dominant. Since λ is antidominant, $\langle e(\omega_{\sigma_{U \cap L}}) - \lambda, \alpha^\vee \rangle = 0$ implies $\langle e(\omega_{\sigma_{U \cap L}}), \alpha^\vee \rangle = \langle \lambda, \alpha^\vee \rangle = 0$. By pairing (3.8) with $\beta^\vee \in \Delta_L^\vee$ and using $\langle e(\omega_\xi), \beta^\vee \rangle = 0$, that λ is antidominant and $e(\omega_{\sigma_{U \cap L}})$ is dominant we obtain that $\langle \lambda, \beta^\vee \rangle = 0$ for all $\beta \in \Delta_L$, and also $\langle \lambda, \alpha^\vee \rangle = 0$ by above. This implies that $\underline{L}(\sigma'_0)^{N_1}$ is trivial on L'_1 by Lemma 3.65. Hence $\sigma'_0 \cong \underline{L}(\sigma'_0)^N$ (an L -stable subspace of $\underline{L}(\sigma'_0)^{N_1}$) is trivial on $L \cap L'_1$. As moreover $\tau|_Z = \xi$ is trivial on $Z \cap L'_\alpha$ we deduce that $\sigma = \sigma_0 \otimes \tau$ is trivial on $Z \cap L'_\alpha$, as desired.

Case 2: μ^{L_β} has a pole at $\xi \delta_B^{-1/2}$ only if $\beta \in \Delta_L$. From the embedding $(\mathrm{n}\text{-Ind}_P^G \tau \delta_P^{-1/2})^{\mathrm{sm}} \hookrightarrow (\mathrm{Ind}_B^G \xi \delta_B^{-1/2})^{\mathrm{sm}}$ we obtain (as in the proof of Lemma 3.51) that $J_{P|\overline{P}}(\tau \delta_P^{-1/2}) J_{\overline{P}|P}(\tau \delta_P^{-1/2}) = J_{B|B'}(\xi \delta_B^{-1/2}) J_{B'|B}(\xi \delta_B^{-1/2})$, where B' is the Borel subgroup of G such that $B' \supset \overline{N}$ and $B' \cap \underline{L} = B \cap \underline{L}$. By [Wal03, IV.3(4)] we get $J_{B|B'}(\xi \delta_B^{-1/2}) J_{B'|B}(\xi \delta_B^{-1/2})$ equals $\prod_\beta \mu^{L_\beta} (\xi \delta_B^{-1/2})^{-1}$ (up to nonzero constant), where the product runs through $\Phi_{\mathrm{red}}(\underline{B}, \underline{A}_Z) \setminus \Phi_{\mathrm{red}}(\underline{B} \cap \underline{L}, \underline{A}_Z)$, which has no zeros by assumption. By [Wal03, IV.1(13)] and the Weyl group regularity of $\omega_\xi \delta_B^{-1/2}$ established above we deduce that $J_{B|B'}$ and $J_{B'|B}$ are regular at $\xi \delta_B^{-1/2}$. Therefore, $J_{P|\overline{P}}$, $J_{\overline{P}|P}$ are regular at $\tau \delta_P^{-1/2}$ and induce an isomorphism $(\mathrm{n}\text{-Ind}_P^G \tau \delta_P^{-1/2})^{\mathrm{sm}} \cong (\mathrm{n}\text{-Ind}_{\overline{P}}^G \tau \delta_P^{-1/2})^{\mathrm{sm}}$. Analogously to Remark 3.23 we deduce that $(\mathrm{n}\text{-Ind}_P^G \tau \delta_P^{-1/2})^{\mathrm{sm}}$ is irreducible. (The Jacquet module of $(\mathrm{Ind}_B^G \xi \delta_B^{-1/2})^{\mathrm{sm}}$ for B is multiplicity-free, as $\xi \delta_B^{-1/2}$ is G -regular. Hence the Jacquet module of $(\mathrm{n}\text{-Ind}_P^G \tau \delta_P^{-1/2})^{\mathrm{sm}}$ for P is multiplicity-free. It follows that $(\mathrm{n}\text{-Ind}_P^G \tau \delta_P^{-1/2})^{\mathrm{sm}}$ has an irreducible socle that is also its irreducible cosocle by the given isomorphism.) \square

Remark 3.67. We know that, by Remark 3.61, there exists an explicit $k_\alpha \in 2\mathbb{Z}_{>0}$ such that $(\mathrm{n}\text{-Ind}_{B \cap L_\alpha}^{L_\alpha} \xi_u \delta_{B \cap L_\alpha}^s)^{\mathrm{sm}}$ reducible for $s \in \mathbb{R}$ implies that $s \in \frac{1}{k_\alpha} \mathbb{Z}$, where we use the notation of the proof. Therefore we can strengthen Theorem 3.33: in part (ii) (resp. (iii)) it suffices to assume that $\langle e(\omega_\sigma|_{A_L}), \alpha^\vee \rangle > -2n_\alpha/k_\alpha$ for any $\alpha \in \Delta \setminus \Delta_L$ (resp. $\langle e(\omega_{\sigma_{U \cap L}}|_S), \alpha^\vee \rangle > -2n_\alpha/k_\alpha$ for any $\alpha \in \Delta$). (Either of these assumptions implies that $\langle e(\omega_{\sigma_{U \cap L}}|_S) - \lambda, \alpha^\vee \rangle > -2n_\alpha/k_\alpha$ for any $\alpha \in \Delta$ and hence that $e(\omega_{\sigma_{U \cap L}}|_S) - \lambda + \rho$ is strictly dominant in the proof. Here we assumed $F = \mathbb{Q}_p$ for the argument, but we remark that the condition $\langle e(\omega_\sigma|_{A_L}), \alpha^\vee \rangle > -2n_\alpha/k_\alpha$ is equivalent for \underline{G} and $\mathrm{Res}_{F/\mathbb{Q}_p} \underline{G}$.)

3.10. Groups of semisimple rank one. In this subsection we establish a more precise irreducibility result when \underline{G} is semisimple of rank 1. Without loss of generality (replacing \underline{G} by $\mathrm{Res}_{F/\mathbb{Q}_p} \underline{G}$) we may and will assume that $F = \mathbb{Q}_p$.

Theorem 3.68. *Assume Assumption 2.14. Suppose that \underline{G} is of semisimple of rank 1 over $F = \mathbb{Q}_p$. Let σ be a finite-dimensional absolutely irreducible continuous representation of Z . Suppose that either σ is \mathfrak{z}_C -simple or that U is abelian. Then $(\mathrm{Ind}_B^G \sigma)^{\mathrm{cts}}$ is absolutely reducible if and only if, after perhaps replacing C with a finite extension, $\underline{L}(\sigma') \in \mathcal{O}^G$.*

We remark that $\sigma \in \mathcal{O}^Z$, after perhaps replacing C by a finite extension, by Lemma 2.49. Also note that the condition $\underline{L}(\sigma') \in \mathcal{O}^G$ is made more explicit in Lemma 2.29.

Proof. Let ω_σ denote the central character of σ . Suppose that $(\mathrm{Ind}_B^G \sigma)^{\mathrm{cts}}$ is reducible. Assume first that $\underline{G}^{\mathrm{der}}$ is simply connected. Then, after perhaps replacing C by a finite extension, we can write $\sigma = \sigma_0 \otimes \tau$ as in §2.8. In particular, σ_0 is \mathfrak{z}_C -simple and $\underline{L}(\sigma'_0)$ is equimaximal with maximal parabolic Q . Then $Q \neq B$ by

Corollary 2.57, i.e. $Q = G$ and $\underline{L}(\sigma'_0) \in \mathcal{O}^G$. By Lemma 2.52 there exists an irreducible subrepresentation π of $(\text{Ind}_B^G \tau)^{\text{sm}}$ that is not dense in $(\text{Ind}_B^G \tau)^{\text{cts}}$. Pick any nonzero element $f \in \pi$. If $f(1) = 0$ and σ is \mathfrak{z}_C -simple, then by smoothness of f and the Bruhat decomposition we get a contradiction from Lemma 2.2 (as $\dim_C \tau = 1$). If $f(1) = 0$ and U is abelian, then pick a compact open subgroup U_0 of U such that $\text{supp}(f) \subset B \backslash Bw_0U_0$ and any U_0 -eigenvector $f' \in \langle U_0 \cdot f \rangle$ (which exists as U_0 is abelian, after perhaps replacing C by a finite extension). Then f' takes values in a 1-dimensional subspace of τ and we get a contradiction from Lemma 2.2. So $f(1) \neq 0$ and then for any $z \in S$ we see that $zf - \omega_\sigma(z)f \in \pi$ vanishes at 1, so again by Lemma 2.2 we deduce that $zf = \omega_\sigma(z)f$ for all $z \in S$. By smoothness of f we deduce that f is fixed by U and \bar{U} , hence by G' . Therefore, for $z \in Z \cap G'$ we have $zf(1) = f(z) = f(1)$, i.e. $f(1) \in \tau^{Z \cap G'}$. As $Z \cap G'$ is normal in Z and $f(1) \neq 0$ we get that $Z \cap G'$ acts trivially on τ . Therefore, τ extends to a smooth representation of G , so $\underline{L}(\sigma') \cong \underline{L}(\sigma'_0) \otimes \tau' \in \mathcal{O}^G$.

For general \underline{G} , as in the proof of Corollary 2.31 we take a z -extension $1 \rightarrow \underline{T} \rightarrow \underline{\tilde{G}} \rightarrow \underline{G} \rightarrow 1$, where $\underline{\tilde{G}}^{\text{der}}$ is simply connected and $1 \rightarrow T \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ on F -points. By pullback to \tilde{G} we obtain $\tilde{B} = \tilde{Z}\tilde{U}$, and by inflation we obtain $\tilde{\sigma} \in \mathcal{O}^{\tilde{Z}}$. As $(\text{Ind}_{\tilde{B}}^{\tilde{G}} \tilde{\sigma})^{\text{cts}}$ is reducible, $\underline{L}(\tilde{\sigma}') \in \mathcal{O}^{\tilde{G}}$ by above. As $\underline{L}(\tilde{\sigma}') \in \mathcal{O}^{\tilde{B}}$ arises by inflation from $\underline{L}(\sigma') \in \mathcal{O}^B$ (in particular T acts trivially on $\underline{L}(\tilde{\sigma}')$), we deduce that $\underline{L}(\sigma') \in \mathcal{O}^G$.

Conversely, suppose that $\underline{L}(\sigma') \in \mathcal{O}^G$ (so finite-dimensional). Then we obtain continuous maps as follows:

$$\begin{aligned} \underline{L}(\sigma')' &= \mathcal{F}_G^G(\underline{L}(\sigma'), 1) \hookrightarrow \mathcal{F}_G^G(\underline{L}(\sigma'), (\text{Ind}_B^G 1)^{\text{sm}}) = \mathcal{F}_B^G(\underline{L}(\sigma'), 1) \\ &\hookrightarrow \mathcal{F}_B^G(\underline{M}(\sigma'), 1) \cong (\text{Ind}_B^G \sigma)^{\text{an}} \hookrightarrow (\text{Ind}_B^G \sigma)^{\text{cts}}, \end{aligned}$$

which proves the reducibility. \square

Remark 3.69. If \underline{G} is one of the groups $\text{SL}_2(D)$, quasisplit SU_3 , or the rank 1 inner form of Sp_4 (see §3.8), then the assumption that σ is \mathfrak{z}_C -simple or that U is abelian is satisfied: in the first and third cases U is abelian, and in the second case Z is abelian (so $\dim_C \tau = 1$, i.e. σ is \mathfrak{z}_C -simple).

In the following corollary we allow arbitrary F/\mathbb{Q}_p .

Corollary 3.70. *Assume Assumption 2.14. Suppose that \underline{G} is split of semisimple rank 1 over F , with unique simple root α . Let $\chi: Z \rightarrow C^\times$ be a continuous character. Then $(\text{Ind}_B^G \chi)^{\text{cts}}$ is absolutely reducible if and only if $\chi \circ \alpha^\vee$ is non-positive algebraic.*

Proof. We first use a z -extension to reduce to the case where $\underline{G}^{\text{der}}$ is simply connected. Moreover, by Proposition 2.42, we may assume $\underline{G} = \underline{G}^{\text{der}} (\cong \text{SL}_2)$. Then $\alpha^\vee: \mathbb{G}_m \rightarrow \underline{Z}$ is an isomorphism. Note that either condition in the corollary is unchanged if we replace C by a finite extension, and we will assume for the rest of the proof that C is sufficiently large without further comment. We set $\tilde{\underline{G}} := \text{Res}_{F/\mathbb{Q}_p} \underline{G}$. If $\chi \circ \alpha^\vee$ is non-positive algebraic, then χ is algebraic and $\underline{L}(\chi^{-1}) \in \mathcal{O}^{\tilde{\underline{G}}}$ by Lemma 2.29. Hence $(\text{Ind}_{\tilde{B}}^{\tilde{\underline{G}}} \chi)^{\text{cts}}$ is reducible by Theorem 3.68. Conversely, if $(\text{Ind}_{\tilde{B}}^{\tilde{\underline{G}}} \chi)^{\text{cts}}$ is reducible, then $\underline{L}(\chi^{-1}) \in \mathcal{O}^{\tilde{\underline{G}}}$ by Theorem 3.68. Hence by Lemma 2.22(i) we know that $\underline{L}(\chi^{-1})$ is algebraic. Therefore $\chi^{-1} \cong \underline{L}(\chi^{-1})^{\tilde{U}}$ is algebraic, and from $L(-d\chi) \in \mathcal{O}^{\tilde{\mathfrak{q}}}$ we deduce that the algebraic character $\chi \circ \alpha^\vee$ is non-positive. \square

APPENDIX A. ORLIK–STRAUCH: THE GENERAL CASE

The goal of this appendix is to generalize the main results of [OS15], [OS14b] to a general connected reductive group. As much as possible we keep the notation of [OS15], [OS14b]. In particular, L (not F) denotes the ground field and K (not C) denotes the coefficient field. Let \underline{G} be a connected reductive group over L . Fix a minimal parabolic subgroup \underline{B} and let $\underline{P} = \underline{L}_{\underline{P}}\underline{U}_{\underline{P}}$ denote a standard parabolic subgroup. Recall the abelian categories $\mathcal{O}^{\mathfrak{p}}$ and \mathcal{O}^P defined in §2.3.

Just as in [OS14b], for $M \in \mathcal{O}^P$ and V an admissible smooth representation of L_P we can then define $\mathcal{F}_P^G(M, V)$ as follows. Pick any finite-dimensional (locally analytic) P -subrepresentation $W \subset M$ that generates M as $U(\mathfrak{g})$ -module. (We use the convention of [OS15], [OS14b] to write $U(\mathfrak{g})$ for $U(\mathfrak{g} \otimes_L K)$.) Then $\mathcal{F}_P^G(M, V)$ is the closed subrepresentation of $(\text{Ind}_P^G W' \otimes V)^{\text{sm}}$ that is annihilated by $\ker(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \rightarrow M)$, cf. [OS14b, §3.8].

Theorem A.1. *Suppose that K is sufficiently large (depending only on \underline{G}), and keep the notation above. The main results of [OS14b] hold for \underline{G} . This means:*

- (i) \mathcal{F}_P^G is functorial and exact in both arguments.
- (ii) If $Q = L_Q U_Q \supset P$, $M \in \mathcal{O}^Q$, V an admissible smooth representation of L_P , then $\mathcal{F}_P^G(M, V) \cong \mathcal{F}_P^G(M, (\text{Ind}_{P \cap L_Q}^{L_Q} V)^{\text{sm}})$.
- (iii) Suppose that $M \in \mathcal{O}^P$ such that
 - (a) \underline{P} is maximal among parabolic subgroups of \underline{G} such that $M \in \mathcal{O}^{\mathfrak{p}}$,
 - (b) M is simple as $U(\mathfrak{g})$ -module,
 and suppose that V is an irreducible (admissible) smooth representation of L_P . Then $\mathcal{F}_P^G(M, V)$ is topologically irreducible.

Remark A.2. Assumption (a) is weaker than saying $\mathfrak{p}_K := \mathfrak{p} \otimes_L K$ is maximal for M because \mathfrak{g}_K can have more parabolic subalgebras than those coming from G when \underline{G} is nonsplit.

Parts (i) and (ii) follow exactly as in [OS15], [OS14b] and the remainder of this appendix will focus on proving part (iii). The basic idea is to deduce (iii) by comparison with the split case, by considering $\underline{G} \times_L L'$ for a carefully chosen finite extension L'/L , like in [Bre16, Appendix] in the case of restrictions of scalars of split groups. (We will need that L' embeds in K , which is why we demand that K be sufficiently large.)

Fix a maximal split torus \underline{S} of \underline{G} over L . Let Φ denote the possibly non-reduced root system of $(\underline{G}, \underline{S})$ and W its Weyl group. Choose any special point x of \underline{G} in the apartment of \underline{S} .

Let $\Phi^+ \subset \Phi$ denote the set of positive roots corresponding to \underline{B} , with simple roots Δ . Choose a “special” subtorus \underline{T}_s of \underline{G} over L containing \underline{S} , i.e. \underline{T}_s becomes a maximal split torus after base change to the maximal unramified extension of L . Let \underline{T} denote the centraliser of \underline{T}_s in \underline{G} , which is a maximal torus of \underline{G} (as \underline{G} becomes quasisplit over the maximal unramified extension of L).

Proposition A.3. *There exists a finite Galois extension L'/L splitting \underline{G} such that x remains special in the building of the split group $\underline{G}' := \underline{G} \times_L L'$.*

Proof. Assume that the valuation ω of L satisfies $\omega(L^\times) \subset \mathbb{Q}$ and extend ω uniquely to an algebraic closure \bar{L} . All extensions of L below will be taken inside \bar{L} . Let $\Phi_{\text{red}} \subset \Phi$ be the subset of reduced roots.

We first suppose \underline{G} quasisplit, with splitting field \tilde{L} . Let $\tilde{\Phi}$ denote the absolute roots. Let φ be the valuation defined by a Chevalley–Steinberg system as in [BT84, §4], so the hyperplanes in the apartment of \underline{S} are given by $\{a(x - \varphi) + r = 0 \mid r \in \Gamma_a, a \in \Phi_{\text{red}}\}$. From [BT84, 4.2.21] we get that Γ_a is an infinite cyclic subgroup

of \mathbb{Q} for all $a \in \Phi_{\text{red}}$, and from [BT84, 4.2.1] that $\Gamma_{\tilde{a}}$ is an infinite cyclic subgroup of \mathbb{Q} for all $\tilde{a} \in \tilde{\Phi}$. Hence there exists $e \in \mathbb{Z}_{>0}$ such that $\Gamma_a \subset \frac{1}{e}\Gamma_{\tilde{a}}$ whenever $\tilde{a}|_{\underline{S}} = a \in \Phi_{\text{red}}$ and $2\Gamma_a \subset \frac{1}{e}\Gamma_{\tilde{b}}$ whenever $\tilde{b}|_{\underline{S}} = 2a \in \Phi$ and $a \in \Phi_{\text{red}}$. Letting L'/\tilde{L} denote any finite extension of ramification index a multiple of e , we get that x is still special over L' (cf. [BT84, 4.2.24]). (In fact, $e = 2$ works always and $e = 1$ works if Φ is reduced.) The same argument shows that if x' lies in the apartment of \underline{S} such that $a(x' - \varphi) \in \mathbb{Q}$ for all $a \in \Phi_{\text{red}}$, then x' becomes special after base change to a suitably ramified extension of \tilde{L} .

For \underline{G} general, let L_0/L be a finite unramified extension with Galois group Γ such that \underline{T}_s becomes split over L_0 ; in particular, $\underline{G} \times_L L_0$ becomes quasplit. We consider the embedding of buildings $\mathcal{B}_L(\underline{G}) \hookrightarrow \mathcal{B}_{L_0}(\underline{G} \times_L L_0)$ (unramified base change). Then the vertex x lies in a unique Γ -invariant facet F in the apartment of \underline{T}_s (inside $\mathcal{B}_{L_0}(\underline{G} \times_L L_0)$), so x equals the average of the vertices of F . By above it then suffices to show that any vertex x' of the apartment of \underline{T}_s satisfies $a(x' - \varphi) \in \mathbb{Q}$ for all $a \in \Phi_{\text{red}}$ (where Φ_{red} denotes the reduced roots of $\underline{G} \times_L L_0$). This is clear: as x' is a vertex we have $a(x' - \varphi) \in \Gamma_a \subset \mathbb{Q}$ for $a \in X \subset \Phi_{\text{red}}$ for some maximal linearly independent subset X , which implies $a(x' - \varphi) \in \mathbb{Q}$ for all $a \in \Phi_{\text{red}}$ (as $\mathbb{Z}X \subset \mathbb{Z}\Phi_{\text{red}}$ has finite index). \square

Let L'/L be a finite Galois extension splitting \underline{G} such that the image x' of the special point x in the building of $\underline{G}' := \underline{G} \times_L L'$ is still special. We assume that K is large enough to contain an embedding $L' \hookrightarrow K$ and we fix such an embedding, extending the given embedding $L \hookrightarrow K$. Let k_L (resp. $k_{L'}$) denote the residue field of L (resp. L').

Let \underline{P} be a (standard) parabolic subgroup of \underline{G} containing \underline{S} with Lie algebra \mathfrak{p} . Let $\underline{U}_{\underline{P}}^-$ be the unipotent radical of the opposite parabolic \underline{P}^- (with respect to \underline{S}) with Lie algebra $\mathfrak{u}_{\underline{P}}^-$.

Let $\underline{S}' := \underline{S} \times_L L'$ and $\underline{T}' := \underline{T} \times_L L'$, so that by construction x' is contained in the apartment of \underline{T}' . Let $\underline{G}'_{x'}$ denote the connected reductive model of \underline{G}' over $\mathcal{O}_{L'}$ defined by x' , and let $\underline{S}'_{x'}$ denote the scheme-theoretic closure of \underline{S}' in $\underline{G}'_{x'}$ (a split torus extending \underline{S}') and similarly define the split torus $\underline{T}'_{x'}$. We define the parabolic subgroup $\underline{P}' := \underline{P} \times_L L'$ of \underline{G}' and its unipotent radical $\underline{U}_{\underline{P}'}^- := \underline{U}_{\underline{P}}^- \times_L L'$. Let also \underline{G}_x denote the connected reductive model of \underline{G} over \mathcal{O}_L defined by x .

Let $G := \underline{G}(L)$, $G' := \underline{G}'(L')$, etc., so G is a closed L -analytic subgroup of G' . Let $G'_0 := \underline{G}'_{x'}(\mathcal{O}_{L'})$ and $G_0 := G \cap G'_0$; these are compact open in G' and G , respectively. Let $P'_0 := P' \cap G'_0$, $U_{P',0}^- := U_{P'}^- \cap G'_0$, $P_0 := P \cap G_0$, $U_{P,0}^- := U_P^- \cap G_0$.

By construction G_0 contains $\underline{G}_x(\mathcal{O}_L)$, so that $G = G_0P$. (Any compact subgroup of G' that fixes x' has to be contained in G'_0 , as \underline{G}' is split.)

Let $\mathfrak{g}'_0 := \text{Lie } \underline{G}'_{x'}$, which is an $\mathcal{O}_{L'}$ -Lie lattice inside $\mathfrak{g}' := \text{Lie } \underline{G}' = \mathfrak{g} \otimes_L L'$ that is stable by the adjoint action of G'_0 . Let $\mathfrak{g}_0 := \mathfrak{g} \cap \mathfrak{g}'_0$, which is an \mathcal{O}_L -Lie lattice inside \mathfrak{g} that is stable by the action of G_0 , and moreover $\mathfrak{g}_0 \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ is of finite index in \mathfrak{g}'_0 . By the algebraic action of $\underline{S}'_{x'}$ on $\mathfrak{g}'_0 = \text{Lie } \underline{G}'_{x'}$ we see that

$$(A.1) \quad \mathfrak{g}'_0 = (\mathfrak{g}'_0 \cap \mathfrak{u}_{\underline{P}'}^-) \oplus (\mathfrak{g}'_0 \cap \mathfrak{p}')$$

as $\mathcal{O}_{L'}$ -modules.

By Lemma A.4 (applied with $\underline{S}'_0 = \underline{S}'_{x'}$, $V = \mathfrak{g}$, $V_1 = \mathfrak{u}_{\underline{P}}^-$, $V_2 = \mathfrak{p}$, $M' = \mathfrak{g}'_0$) we also have

$$(A.2) \quad \mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{u}_{\underline{P}}^-) \oplus (\mathfrak{g}_0 \cap \mathfrak{p})$$

as \mathcal{O}_L -modules; we even see that $\mathfrak{g}_0 \cap \mathfrak{u}_{\underline{P}}^- = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_{\underline{P}}^-} (\mathfrak{g}_0 \cap \mathfrak{u}_{\alpha})$.

Lemma A.4. *Suppose $A \subset B$ are integral domains with fields of fractions $E \subset F$.*

- (i) Suppose \underline{S} is a split torus over E , \underline{S}'_0 a split torus over B , and we are given an isomorphism $\underline{S} \times_E F \cong \underline{S}'_0 \times_B F$. Then there exists a unique split torus \underline{S}_0 over A together with isomorphisms $\underline{S}_0 \times_A E \cong \underline{S}$, $\underline{S}_0 \times_A B \cong \underline{S}'_0$ compatible with the isomorphism above after base change to F .
- (ii) Keep the notation as in (i). Suppose $V = V_1 \oplus V_2$ is an isomorphism of finite-dimensional \underline{S} -modules and suppose that M' is an \underline{S}'_0 -module together with an isomorphism $M' \otimes_B F \cong V \otimes_E F$ that is compatible with the actions of $\underline{S}'_0 \times_B F \cong \underline{S} \times_E F$. Let $M := M' \cap V$. If for all $\chi \in X^*(\underline{S})$ we have $(V_1)_\chi = 0$ or $(V_2)_\chi = 0$, then $M = (M \cap V_1) \oplus (M \cap V_2)$.

Proof. (i) We recall from [Jan03, §I.2.5] that over an integral domain A we have an (anti-)equivalence of categories between abelian groups and diagonalizable group schemes over A , and this equivalence is moreover compatible with base change $A \rightarrow B$. Thus the claim becomes obvious.

(ii) Let $X := X^*(\underline{S}_0)$, so that we may identify $\underline{S}_0 \times_A C$ with the spectrum of the Hopf algebra $C[X]$ for any map of integral domains $A \rightarrow C$. Let $\Delta_V: V \rightarrow V \otimes_E E[X]$ denote the comodule corresponding to V and $\Delta_{M'}: M' \rightarrow M' \otimes_B B[X]$ the comodule corresponding to M' , so that Δ_V and $\Delta_{M'}$ become identified after base change to F . It follows that Δ_V sends M to $(V \otimes_E E[X]) \cap (M' \otimes_B B[X])$ inside $V \otimes_E F[X]$. Hence $\Delta_V(M) \subset M \otimes_A A[X]$, i.e. M becomes an \underline{S} -module whose base change to B (resp. E) is identified with M' (resp. V). Therefore, $M = \bigoplus_{\chi \in X} M_\chi$ [Jan03, §I.2.11]. For any $\chi \in X$ we have $M_\chi \subset V_\chi = (V_1)_\chi \oplus (V_2)_\chi = (V_i)_\chi$ for some i , by assumption, so $M_\chi \subset M \cap V_i$ and we are done. \square

Recall the equivalence between uniform pro- p groups H and powerful \mathbb{Z}_p -Lie algebras \mathfrak{h} [DdSMS99, Theorem 9.10]. If \mathfrak{h} is a powerful \mathbb{Z}_p -Lie algebra, then the corresponding uniform pro- p group is defined by the convergence of the Baker–Campbell–Hausdorff formula [DdSMS99, Theorem 9.8], and we denote it by $\text{BCH}(\mathfrak{h})$. It is locally \mathbb{Q}_p -analytic group with Lie algebra $\mathfrak{h} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $\mathfrak{h}' \subset \mathfrak{h}$ is a saturated sub- \mathbb{Z}_p -Lie algebra of \mathfrak{h} , then \mathfrak{h}' is also powerful and $\text{BCH}(\mathfrak{h}') \subset \text{BCH}(\mathfrak{h})$ is a closed subgroup [DdSMS99, Scholium to Theorem 9.10]. Conversely, if H is a uniform pro- p group, then it is (uniquely) a locally \mathbb{Q}_p -analytic group and the \mathbb{Z}_p -lattice $\log(H) \subset \text{Lie}(H)$ is the corresponding powerful \mathbb{Z}_p -Lie algebra. (Here, \log denotes the functorial logarithm map of a locally analytic group, which is defined on the union of all compact subgroups [Bou72, III.7.6].) An L -uniform group is a uniform pro- p group H together with an \mathcal{O}_L -Lie algebra structure on $\log(H)$ that extends the given \mathbb{Z}_p -Lie algebra structure.

Lemma A.5. *Any L -uniform group H is (uniquely) a locally L -analytic group.*

Proof. The Baker–Campbell–Hausdorff formula does not depend on the base field, so it converges on the \mathcal{O}_L -Lie algebra $\log(H)$, making it into a locally L -analytic group with Lie algebra $\text{Lie}(H)$ [Sch11, §17]. The uniqueness follows exactly as in the proof of [Sch11, Theorem 29.8]. \square

This lemma implies that our definition agrees with the one in [OS10, Remark 2.2.5]. (If H is an open uniform subgroup of a locally L -analytic group G such that $\log(H) \subset \text{Lie}(G)$ is \mathcal{O}_L -stable, then we give $\log(H)$ the induced \mathcal{O}_L -Lie algebra structure. Conversely, if H is L -uniform in our sense, then we take $G := H$ by Lemma A.5.) We get an equivalence between L -uniform groups (with locally L -analytic morphisms) and powerful \mathcal{O}_L -Lie algebras (meaning that the underlying \mathbb{Z}_p -Lie algebra is powerful).

If \mathfrak{h} is a powerful \mathbb{Z}_p -Lie lattice in \mathfrak{g} , then so is $p^n \mathfrak{h}$ for any $n \geq 0$. For n sufficiently large there exists a (non-canonical!) open embedding $i: \text{BCH}(p^n \mathfrak{h}) \hookrightarrow$

G_0 of locally analytic groups whose associated map on Lie algebras is the identity [Sch11, Proposition 18.17].

Lemma A.6. *If $\ell \geq \kappa$, then $G'_{(\ell)} := \ker(\underline{G}'_{x'}(\mathcal{O}_{L'}) \rightarrow \underline{G}'_{x'}(\mathcal{O}_{L'}/p^\ell))$ is a uniform pro- p group, and we have $\log(G'_{(\ell)}) = p^\ell \mathfrak{g}'_0$ inside \mathfrak{g}' .*

Proof. For any smooth group scheme \underline{H} over $\mathcal{O}_{L'}$ let $H_{(\ell)} := \ker(\underline{H}(\mathcal{O}_{L'}) \rightarrow \underline{H}(\mathcal{O}_{L'}/p^\ell))$. Let Φ' denote the roots of $(\underline{G}', \underline{T}')$ and fix some subset of positive roots Φ'^+ . Then $G'_{(\ell)} = G'_{x',(\ell)}$ is contained in the Iwahori subgroup defined by the Borel subgroup of $\underline{G}'_{x'} \times_{\mathcal{O}_{L'}} k_{L'}$ that contains $\underline{T}'_{x'} \times_{\mathcal{O}_{L'}} k_{L'}$ and corresponds to Φ'^+ , hence

$$G'_{x',(\ell)} \subset \prod_{a \in \Phi'^-} \underline{U}'_{a,x'}(\mathcal{O}_{L'}) \cdot \underline{T}'_{x'}(\mathcal{O}_{L'}) \cdot \prod_{a \in \Phi'^+} \underline{U}'_{a,x'}(\mathcal{O}_{L'}),$$

where $\underline{U}'_{a,x'}$ is the scheme-theoretic closure of the root subgroup U'_a inside $\underline{G}'_{x'}$, and we fixed some ordering of the roots. As $\prod_{a \in \Phi'^-} \underline{U}'_{a,x'} \times \underline{T}'_{x'} \times \prod_{a \in \Phi'^+} \underline{U}'_{a,x'} \rightarrow \underline{G}'_{x'}$ is an open immersion, it follows that

$$G'_{x',(\ell)} = \prod_{a \in \Phi'^-} U'_{a,x',(\ell)} \cdot T'_{x',(\ell)} \cdot \prod_{a \in \Phi'^+} U'_{a,x',(\ell)}.$$

By explicit calculation for $\underline{U}'_{a,x'} \cong \mathbb{G}_a$ and $\underline{T}'_{x'} \cong \mathbb{G}_m^r$ we see that $\log(U'_{a,x',(\ell)}) = p^\ell \text{Lie } \underline{U}'_{a,x'}$ and $\log(T'_{x',(\ell)}) = p^\ell \text{Lie } \underline{T}'_{x'}$.

To justify that $G'_{x',(\ell)}$ is uniform, it suffices to show it is a standard \mathbb{Q}_p -analytic group [DdSMS99, Theorem 8.31]. Fix any \mathbb{Z} -basis $\lambda_1, \dots, \lambda_r$ of $X_*(\underline{T}'_{x'})$. By [BT84, §3.2.13, §4.6.15] there exist $\mathcal{O}_{L'}$ -isomorphisms $x_a: \mathbb{G}_a \xrightarrow{\sim} \underline{U}'_{a,x'}$ ($a \in \Phi'$) defining a Chevalley system as in [BT84, §3.2.2]. Then we get an topological isomorphism

$$\begin{aligned} \theta: (p^\ell \mathcal{O}_{L'})^{\Phi'} \times (p^\ell \mathcal{O}_{L'})^r &\xrightarrow{\sim} G'_{x',(\ell)} \\ (u_a, t_i) &\mapsto \prod_{a \in \Phi'^-} x_a(u_a) \prod_{i=1}^r \lambda_i(1+t_i) \prod_{a \in \Phi'^+} x_a(u_a). \end{aligned}$$

By the commutation relations in [BT84, §3.2.3], the relation $x_a(u)x_{-a}(v) = x_{-a}(-v(1-uv)^{-1})a^\vee(1-uv)x_a(u(1-uv)^{-1})$ (which follows from [BT84, §3.2.1]), as well as $\lambda_i(1+t)x_a(u) = x_a((1+t)^{\langle a, \lambda_i \rangle} u) \lambda_i(1+t)$ it follows that θ is a global chart making $G'_{x',(\ell)}$ into a standard group and hence $G'_{x',(\ell)}$ is uniform.

By using this as well as [DdSMS99, Theorem 4.17] with a minimal topological generating set contained in $\bigcup_{a \in \Phi'} U'_{a,x',(\ell)} \cup T'_{x',(\ell)}$ we deduce $\log(G'_{x',(\ell)}) = p^\ell \mathfrak{g}'_0$, as required. (Note that [DdSMS99, Theorem 4.17] shows that if g_1, \dots, g_r is a minimal topological generating set of a uniform group H , then $\log(H) = \bigoplus_i \mathbb{Z}_p \log(g_i)$, and note also that \log is functorial.) \square

Let $\kappa := 1$ if $p > 2$ and $\kappa := 2$ if $p = 2$. For $m_0 \geq \kappa$ the $\mathcal{O}_{L'}$ -Lie lattice $p^{m_0} \mathfrak{g}'_0$ in \mathfrak{g}' is powerful. Define L' -uniform groups $H' := \text{BCH}(p^{m_0} \mathfrak{g}'_0)$, $H'^- := \text{BCH}(p^{m_0} \mathfrak{g}'_0 \cap \mathfrak{u}_{P'}^-)$, $H'^+ := \text{BCH}(p^{m_0} \mathfrak{g}'_0 \cap \mathfrak{p}'^+)$. By Lemma A.6 and the equivalence between uniform groups and powerful \mathbb{Z}_p -Lie algebras we get a *unique* open embedding $H' := \text{BCH}(p^{m_0} \mathfrak{g}'_0) \hookrightarrow G'_0$ whose image is $G'_{(m_0)}$ and whose derivative is the identity. For any $\gamma \in G'_0$, the conjugation action of γ on G'_0 induces the adjoint action of γ on $p^{m_0} \mathfrak{g}'_0$. Note that H'^-, H'^+ are closed subgroups of H' . For m_0 sufficiently large we have $H'^- \subset U_{P',0}^-$ and $H'^+ \subset P'_0$ because this true on the level of Lie algebras. (Actually, $m_0 \geq \kappa$ suffices, by a variant of Lemma A.6.) By (A.1) and [DdSMS99, Theorem 9.8] we see that the multiplication map $H'^- \times H'^+ \rightarrow H'$ is a topological isomorphism and hence that $H'^- = H' \cap U_{P'}^-$, $H'^+ = H' \cap P'$. From $H' \triangleleft G'_0$ it then follows that $H'^- \triangleleft U_{P',0}^-$ and $H'^+ \triangleleft P'_0$.

The closed subgroup $H := \text{BCH}(p^{m_0}\mathfrak{g}_0)$ of H' is contained in G for m_0 sufficiently large, because this is true on the level of Lie algebras. As $G_0 = G \cap G'_0$ normalizes $\mathfrak{g}_0 = \mathfrak{g} \cap \mathfrak{g}'_0$, it follows that $H \triangleleft G_0$. Letting $H^- := \text{BCH}(p^{m_0}\mathfrak{g}_0 \cap \mathfrak{u}_P^-)$, $H^+ := \text{BCH}(p^{m_0}\mathfrak{g}_0 \cap \mathfrak{p})$ we see that $H^- \triangleleft U_{P,0}^-$, $H^+ \triangleleft P_0$. By (A.2), the multiplication map $H^- \times H^+ \rightarrow H$ is a topological isomorphism and hence $H^- = H \cap U_P^-$, $H^+ = H \cap P$. By construction, H, H^-, H^+ are L -uniform groups.

For $m \geq 0$ let $H^m := P_{m+1}(H) = \text{BCH}(p^{m+m_0}\mathfrak{g}_0) \triangleleft G_0$ (where $(P_m(H))_{m \geq 1}$ is the lower p -series, cf. [DdSMS99, Definition 1.15]) and likewise $H^{-,m} \triangleleft U_{P,0}^-$, $H^{+,m} \triangleleft P_0$, $H'^m \triangleleft G'_0$, $H'^{-,m} \triangleleft U_{P',0}^-$. These are all L - (resp. L' -) uniform groups. Note that by construction \mathfrak{g}_0 is an \mathcal{O}_L -direct summand of \mathfrak{g}'_0 . In particular, H^m is a closed subgroup of H'^m which is topologically a direct factor (an ordered basis of H^m as a uniform group can be extended to an ordered basis of H'^m).

For any $r \in p^{\mathbb{Q}} \cap (p^{-1}, 1)$ we recall that we have a continuous algebra (semi)norm $|\cdot|_r$ on the locally analytic distribution algebra $D(H)$, defined by the uniform structure of H [OS10, 2.2.6]. Its completion $D_r(H)$ (or more precisely $D_r^{(L)}(H)$) is a noetherian Banach algebra. Let $U_r(\mathfrak{g}) = U_r(\mathfrak{g}, H)$ denote the closure of $U(\mathfrak{g})$ in $D_r^{(L)}(H)$ and $U_r(\mathfrak{g}') = U_r(\mathfrak{g}', H')$ the closure of $U(\mathfrak{g}')$ in $D_r^{(L')}(H')$. Then $D_r(H)$ is free as left and right $U_r(\mathfrak{g})$ -module, admitting a basis in $K[H]$, and $U_r(\mathfrak{g})$ is noetherian [Koh07, Theorem 1.4.2].

Corollary A.7. *Suppose $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$. Then we have a commutative diagram*

$$\begin{array}{ccc} U(\mathfrak{g}) & \xlongequal{\quad} & U(\mathfrak{g}') \\ \downarrow & & \downarrow \\ U_r(\mathfrak{g}) & \dashrightarrow & U_r(\mathfrak{g}') \end{array}$$

where the bottom map is a morphism of Banach algebras. Moreover, all maps are equivariant for the adjoint action of $G_0 \subset G'_0$.

Proof. Let $s := r^{p^m}$. By density the bottom map is unique, if it exists, and it is automatically an algebra homomorphism and G_0 -equivariant. It thus suffices to consider the underlying Banach spaces of the bottom row.

Note that $p^{m_0}\mathfrak{g}_0 \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$ is powerful and we let $H'' := \text{BCH}(p^{m_0}\mathfrak{g}_0 \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}) \subset H'$ (another L' -uniform group, open inside H'). Then $U_r(\mathfrak{g}, H) = U_r(\mathfrak{g}', H'')$ by [Sch08, §5, §6] (taking the closure of $U(\mathfrak{g})$ in $D_s^{(L)}(H^m) \subset D_r^{(L)}(H)$, resp. of $U(\mathfrak{g}')$ in $D_s^{(L')}(H''^{p^m}) \subset D_r^{(L')}(H'')$). We may thus work over L' and it suffices to show that if $H^{(1)} \subset H^{(2)}$ are L' -uniform open subgroups of G' with corresponding powerful $\mathcal{O}_{L'}$ -Lie lattices $\mathfrak{h}_0^{(1)} \subset \mathfrak{h}_0^{(2)}$ in \mathfrak{g}' , we get a morphism of Banach spaces $U_r(\mathfrak{g}', H^{(1)}) \rightarrow U_r(\mathfrak{g}', H^{(2)})$ (compatible with maps from $U(\mathfrak{g}')$).

Pick $(X_i)_{i=1}^d$ an ordered $\mathcal{O}_{L'}$ -basis of $p^m\mathfrak{h}_0^{(2)}$ such that $(Y_i := \varpi_{L'}^{\ell_i} X_i)_{i=1}^d$ is an ordered $\mathcal{O}_{L'}$ -basis of $p^m\mathfrak{h}_0^{(1)}$ (with $\ell_i \in \mathbb{Z}_{\geq 0}$). By [Sch08, §5, §6] we have

$$\begin{aligned} U_r(\mathfrak{g}', H^{(2)}) &= \left\{ \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} d_\beta \mathbf{X}^\beta : d_\beta \in K, |d_\beta| s^{\kappa|\beta|} \rightarrow 0 \right\}, \\ U_r(\mathfrak{g}', H^{(1)}) &= \left\{ \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} e_\beta \mathbf{Y}^\beta : e_\beta \in K, |e_\beta| s^{\kappa|\beta|} \rightarrow 0 \right\} \end{aligned}$$

and the topology can be defined by norms

$$\left\| \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} d_\beta \mathbf{X}^\beta \right\|_s = \sup_{\beta} |d_\beta| s^{\kappa|\beta|}, \quad \left\| \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} e_\beta \mathbf{Y}^\beta \right\|_s = \sup_{\beta} |e_\beta| s^{\kappa|\beta|}.$$

Taking the inclusion, we get a norm-decreasing map of Banach spaces $U_r(\mathfrak{g}', H^{(1)}) \rightarrow U_r(\mathfrak{g}', H^{(2)})$ that is compatible with maps from $U(\mathfrak{g}')$, as required. \square

Let $D(\mathfrak{g}, P_0)$ be the subring of $D(G_0)$ generated by $U(\mathfrak{g})$ and $D(P_0)$. Then any object of \mathcal{O}^P becomes a $D(\mathfrak{g}, P_0)$ -module. We define a continuous algebra norm $|\cdot|_r$ on $D(\tilde{H})$ for any subgroup $H \leq \tilde{H} \leq G_0$ by using $|\cdot|_r$ on $D(H)$ [OS15, (5.5.5)]. Let $D_r(\tilde{H})$ denote its completion. Likewise we define $|\cdot|_r$ on $D(P_0)$ by using $|\cdot|_r$ on $D(H^+)$, and let $D_r(P_0)$ denote its completion, which is a closed subring of $D_r(G_0)$. Let $D_r(\mathfrak{g}, P_0)$ be the subring of $D_r(G_0)$ generated by $U_r(\mathfrak{g})$ and $D_r(P_0)$. Then $D_r(\mathfrak{g}, P_0)$ is a finitely generated $U_r(\mathfrak{g})$ -module, hence a closed subring and noetherian. Let $U_r(\mathfrak{u}_{\bar{P}}) := U_r(\mathfrak{u}_{\bar{P}}, H^-)$ and $U_r(\mathfrak{p}) := U_r(\mathfrak{p}, H^+)$.

Lemma A.8. *We have canonical isomorphisms of Banach spaces:*

$$(A.3) \quad D_r(H) \cong D_r(H^-) \widehat{\otimes} D_r(H^+),$$

$$(A.4) \quad U_r(\mathfrak{g}) \cong U_r(\mathfrak{u}_{\bar{P}}) \widehat{\otimes} U_r(\mathfrak{p}),$$

$$(A.5) \quad D_r(HP_0) \cong D_r(H^-) \widehat{\otimes} D_r(P_0),$$

$$(A.6) \quad D_r(\mathfrak{g}, P_0) \cong U_r(\mathfrak{u}_{\bar{P}}) \widehat{\otimes} D_r(P_0).$$

Proof. As $H \triangleleft G_0$ is open we have $HP_0 = H^-P_0$ with open subgroup $H = H^-H^+$ (i.e. both having Iwahori decomposition with respect to $U_{\bar{P}} \times P \hookrightarrow G$). The proof of (A.3) and (A.5) is exactly as in [OS10, Proposition 3.3.4], with H playing the role of I_0 and HP_0 the role of I . We note that these isomorphisms are induced by convolution, as follows by comparison with [ST05, Remark A.4]. Then (A.4) follows from (A.3). For (A.6), note using (A.5) that the right-hand side is contained in the left-hand side, as $D_r(\mathfrak{g}, P_0)$ is a closed subring of $D_r(HP_0)$ (in turn a closed subring of $D_r(G_0)$). As $D_r(P_0) = U_r(\mathfrak{p})K[P_0]$ we have $D_r(\mathfrak{g}, P_0) = U_r(\mathfrak{g})K[P_0]$, which is contained in the right-hand side by (A.4). \square

Lemma A.9. *Suppose that W is a finite-dimensional \mathfrak{p}_K -module. If $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$ and r is sufficiently close to 1, then we have $U_r(\mathfrak{p}) \widehat{\otimes}_{U(\mathfrak{p})} W \cong W$.*

Proof. By [Sch11, Proposition 18.17] there exists a compact open subgroup P_{00} of P such that W lifts to a locally analytic representation of P_{00} . Take r sufficiently close to 1 so that $H^{+,m} \subset P_{00}$. Note that $U_r(\mathfrak{p}) = D_s(H^{+,m}) \subset D_r(H^+)$ and that W is a coadmissible $D(H^{+,m})$ -module, where $s := r^{p^m}$. We have

$$W \rightarrow U_r(\mathfrak{p}) \widehat{\otimes}_{U(\mathfrak{p})} W \rightarrow D_s(H^{+,m}) \widehat{\otimes}_{D(H^{+,m})} W$$

whose composition is an isomorphism for all r sufficiently close to 1, by coadmissibility of W . But the first map has dense image, hence is surjective (as the image is finite-dimensional, hence complete), so both maps are bijective. \square

Fix again $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$ and r sufficiently close to 1. Recall that \mathfrak{m}_r is defined to be the $U_r(\mathfrak{g})$ -submodule of $D_r(HP_0) \otimes_{D(\mathfrak{g}, P_0)} M$ generated by M . It is finitely generated and hence carries a canonical topology such that the image of M in \mathfrak{m}_r is dense. As in [OS15, Sublemma 5.6] it follows that $\mathfrak{m}_r \cong D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} M$.

Lemma A.10. *Assume $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$ and r sufficiently close to 1. Then $\mathfrak{m}_r \cong U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \cong U_r(\mathfrak{u}_{\bar{P}}) \otimes_{U(\mathfrak{u}_{\bar{P}})} M$ (with canonical topologies).*

Proof. We prove this more generally for any $M \in \mathcal{O}^P$, where we define \mathfrak{m}_r to be $D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} M$. Then \mathfrak{m}_r is a finitely generated $D_r(\mathfrak{g}, P_0)$ -module (hence also a finitely generated $U_r(\mathfrak{g})$ -module) and we endow it with its canonical topology. To show that the natural maps $U_r(\mathfrak{u}_{\bar{P}}) \otimes_{U(\mathfrak{u}_{\bar{P}})} M \rightarrow U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M \rightarrow \mathfrak{m}_r$ are isomorphisms, we use a finite presentation $\bigoplus_{i=1}^n U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W_i \rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \rightarrow M \rightarrow 0$ in \mathcal{O}^P , where W and W_i are finite-dimensional locally analytic representations of P that are direct sums of absolutely simple $\mathfrak{l}_{P,K}$ -modules, to reduce to the case where $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$. Then, using (A.4) of Lemma A.8 (and canonical $U_r(\mathfrak{g})$ -topologies) as well as Lemma A.9, we get

$$\begin{aligned} U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M &\cong U_r(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \cong U_r(\mathfrak{g}) \widehat{\otimes}_{U(\mathfrak{p})} W \\ &\cong U_r(\mathfrak{u}_{\bar{P}}) \widehat{\otimes} (U_r(\mathfrak{p}) \widehat{\otimes}_{U(\mathfrak{p})} W). \\ &\cong U_r(\mathfrak{u}_{\bar{P}}) \otimes W. \end{aligned}$$

By comparing this with

$$\begin{aligned} \mathfrak{m}_r &= D_r(\mathfrak{g}, P_0) \otimes_{D(\mathfrak{g}, P_0)} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W = D_r(\mathfrak{g}, P_0) \otimes_{D(P_0)} W \\ &= D_r(\mathfrak{g}, P_0) \widehat{\otimes}_{D(P_0)} W \\ &= (U_r(\mathfrak{u}_{\bar{P}}) \widehat{\otimes} D_r(P_0)) \widehat{\otimes}_{D(P_0)} W \\ &= U_r(\mathfrak{u}_{\bar{P}}) \widehat{\otimes} (D_r(P_0) \widehat{\otimes}_{D(P_0)} W) \\ &= U_r(\mathfrak{u}_{\bar{P}}) \otimes W \end{aligned}$$

(using (A.6) of Lemma A.8) we deduce the claim. \square

Lemma A.11. *Suppose V is a Banach space over K , \mathcal{H} a finitely generated commutative subalgebra of $\text{End}_{\text{cts}}(V)$, such that every $v \in V$ can be written as a convergent series $v = \sum_{\lambda} v_{\lambda}$ of λ -eigenvectors (with respect to the filter of cofinal subsets of $\text{Hom}_{\text{alg}}(\mathcal{H}, K)$). If $M \subset V$ is a dense \mathcal{H} -stable subspace such that $M = \bigoplus_{\lambda} M_{\lambda}$ and M_{λ} is finite-dimensional for all λ , then we have an order-preserving bijection*

$$\{\mathcal{H}\text{-invariant closed subspaces of } V\} \longleftrightarrow \{\mathcal{H}\text{-invariant subspaces of } M\}$$

given by intersecting with M , respectively taking closure in V .

Remark A.12. In particular, $M_{\lambda} = V_{\lambda}$ for all λ . Also, if W is an \mathcal{H} -invariant closed subspace of V , then $M \cap W \subset W$ and $M/(M \cap W) \subset V/W$ also satisfy the conditions in the lemma. (Note that if $\sum_{\lambda} v_{\lambda} \in W$, then $v_{\lambda} \in W$ for any λ , by [FdL99, Kor. 1.3.8].)

Proof. By the assumptions on V , [FdL99, Kor. 1.3.12] gives uniqueness of the representation $v = \sum_{\lambda} v_{\lambda}$ and continuous projection maps $\pi_{\lambda} : V \rightarrow V_{\lambda}$, $v \mapsto v_{\lambda}$. As $M = \bigoplus_{\lambda} M_{\lambda}$ is dense in V and M_{λ} is finite-dimensional, $\pi_{\lambda}(M) = \pi_{\lambda}(M_{\lambda})$ has to equal V_{λ} , i.e. $M_{\lambda} = V_{\lambda}$. Therefore, the last claim follows from the last item of [FdL99, Kor. 1.3.12]. \square

Theorem A.13. *Assume [OS14b, Assumption 4.1] for the absolute root system. Let $M \in \mathcal{O}^P$ be such that*

- (i) P is maximal among parabolic subgroups of G such that $M \in \mathcal{O}^P$, and
- (ii) M is simple as $U(\mathfrak{g})$ -module.

Assume $r^{p^m} \in p^{\mathbb{Q}} \cap (p^{-1}, p^{-1/\kappa(p-1)})$ for some $m \geq 0$ and r sufficiently close to 1, so in particular $\mathfrak{m}_r \neq 0$. Then for every $g \in G_0$ the $U_r(\mathfrak{g})$ -module $\delta_g \star \mathfrak{m}_r$ is simple, and if $\delta_{g_1} \star \mathfrak{m}_r \cong \delta_{g_2} \star \mathfrak{m}_r$ as $U_r(\mathfrak{g})$ -modules for some $g_i \in G_0$, then $g_1 H^{-\cdot, m} P_0 = g_2 H^{-\cdot, m} P_0$.

This generalizes [OS14b, Theorem 4.7].

Proof. We first show that \mathfrak{m}_r is simple (which implies that $\delta_g \star \mathfrak{m}_r$ is simple), following the proof in [OS14b, Theorem 4.5]. For this it will suffice to show that the map $M \rightarrow \mathfrak{m}_r = U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$ is injective and satisfies the hypotheses of Lemma A.11 with $V := \mathfrak{m}_r$ and $\mathcal{H} := U(\mathfrak{a}_P)$, where \underline{A}_P is the maximal split subtorus of the center of \underline{L}_P (because then every (closed) $\overline{U}_r(\mathfrak{g})$ -submodule of \mathfrak{m}_r corresponds to a $U(\mathfrak{g})$ -submodule of M). We will show this claim is true more generally for any $M \in \mathcal{O}^p$.

Suppose first that $M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \in \mathcal{O}^p$ for some W , a finite-dimensional \mathfrak{p}_K -module that is a direct sum of absolutely simple $\mathfrak{l}_{P,K}$ -modules. As in the proof of Lemma A.10, $\mathfrak{m}_r \cong U_r(\mathfrak{u}_{\overline{P}}) \otimes W$ as $U_r(\mathfrak{u}_{\overline{P}})$ -module (with canonical Banach topology), so $M \rightarrow \mathfrak{m}_r$ is identified with the injection $U(\mathfrak{u}_{\overline{P}}) \otimes W \hookrightarrow U_r(\mathfrak{u}_{\overline{P}}) \otimes W$. Note that $U(\mathfrak{a}_P)$ acts by continuous endomorphisms on \mathfrak{m}_r and that the $U(\mathfrak{a}_P)$ -eigenspaces of M are finite-dimensional (as the roots $\Phi^- \setminus \Phi_{\overline{P}}^-$ of $\mathfrak{u}_{\overline{P}}$ remain non-trivial on \mathfrak{a}_P). On the other hand, $U_r(\mathfrak{u}_{\overline{P}}) = D_s(H^{-\cdot, m})$ ($s = r^{p^m}$) has the description

$$U_r(\mathfrak{u}_{\overline{P}}) = \left\{ \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} d_\beta \mathbf{X}^\beta : d_\beta \in K, |d_\beta| s^{\kappa|\beta|} \rightarrow 0 \right\}$$

by [Sch08, §5], where X_1, \dots, X_d is an ordered \mathcal{O}_L -basis of $\log(H^{-\cdot, m}) = p^{m+m_0} \mathfrak{g}_0 \cap \mathfrak{u}_{\overline{P}}^- = \bigoplus_{\alpha \in \Phi^- \setminus \Phi_{\overline{P}}^-} (p^{m+m_0} \mathfrak{g}_0 \cap \mathfrak{u}_\alpha)$ consisting of \mathfrak{a}_P -eigenvectors. Moreover the norm is given by

$$\left\| \sum_{\beta \in \mathbb{Z}_{\geq 0}^d} d_\beta \mathbf{X}^\beta \right\|_s = \sup_{\beta} |d_\beta| s^{\kappa|\beta|}.$$

By Lemma 2.5, \mathfrak{a}_P acts diagonalizably on W , hence every vector of \mathfrak{m}_r is a convergent sum of \mathfrak{a}_P -eigenvectors.

For general $M \in \mathcal{O}^p$ take a finite-dimensional \mathfrak{p}_K -submodule $W \subset M$ giving an exact sequence $0 \rightarrow \mathfrak{d} \rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \rightarrow M \rightarrow 0$ in \mathcal{O}^p . Tensoring with $U_r(\mathfrak{g})$ gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{d} & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U_r(\mathfrak{g})\mathfrak{d} & \longrightarrow & U_r(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W & \longrightarrow & \mathfrak{m}_r \longrightarrow 0 \end{array}$$

where the middle vertical arrow is injective by above, and the left square is Cartesian by Lemma A.11 (noting that $U_r(\mathfrak{g})\mathfrak{d}$ is the closure of \mathfrak{d}), hence the right vertical arrow is injective. By Remark A.12 we see that Lemma A.11 applies to $M \hookrightarrow \mathfrak{m}_r$.

Now assume that $\delta_{g_1} \star \mathfrak{m}_r \cong \delta_{g_2} \star \mathfrak{m}_r$ as $U_r(\mathfrak{g})$ -modules for some $g_i \in G_0$. Following the beginning of the proof of [OS14b, Theorem 4.7] we will first reduce to the case where $g_1 = 1$, $g_2 \in U_{P,0}^-$. Let I_0 be the Iwahori that is the inverse image under $\underline{G}_x(\mathcal{O}_L) \rightarrow \underline{G}_x^{\text{red}}(k_L)$ of \overline{B}_x , where \overline{B}_x is the minimal parabolic defined by Φ^+ in the reductive quotient $\underline{G}_x^{\text{red}}$ of $\underline{G}_x \times_{\mathcal{O}_L} k_L$. Then we get an Iwahori–Bruhat decomposition $\underline{G}_x(\mathcal{O}_L) = \coprod_{w \in W/W_I} I_0 \dot{w} (P \cap \underline{G}_x(\mathcal{O}_L))$ using the argument of [AHHV17, §IV.5], where $I \subset \Delta$ is the subset corresponding to \underline{P} , W_I is the subgroup of W generated by reflections s_α for $\alpha \in I$, and where \dot{w} denotes a choice of representative in $\underline{G}_x(\mathcal{O}_L) \cap N_G(S)$. From $\underline{G}_x(\mathcal{O}_L) \subset G_0$ and $G = \underline{G}_x(\mathcal{O}_L)P$

we have $G_0 = \underline{G}_x(\mathcal{O}_L)P_0$, so $G_0 = \prod_{w \in W/W_I} I_0 \dot{w} P_0$. As in the proof of [OS14b, Theorem 4.7] we reduce to $g_1 = \dot{w}$ and $g_2 = u$ for some $w \in W$, $u \in U_P^- \cap I_0 \subset U_{P,0}^-$.

Let now Q' (containing \underline{P}') denote the parabolic subgroup of \underline{G}' that is maximal subject to $M \in \mathcal{O}^{\mathfrak{q}'}$. Choose a set of simple roots Δ' of Φ' such that $\Delta \subset \Delta'|_{\underline{S}} \subset \Delta \cup \{0\}$ and let $\Delta'_0 \subset \Delta'$ be the subset of simple roots restricting trivially to \underline{S} . Recall that Δ' is endowed with the $*$ -action of $\Gamma := \text{Gal}(L'/L)$. Let J' (resp. I') denote the subset of Δ' corresponding to Q' (resp. \underline{P}'). The maximality conditions on P (resp. Q') imply that $I' = \bigcap_{\gamma \in \Gamma} \gamma(J')$. Then $I = (I'|_{\underline{S}}) \setminus \{0\}$ as subset of Δ .

Assuming $w \notin W_I$, by Step 1 of [OS14b, Theorem 4.7] there exists a reduced relative root $\beta \in \Phi^+ \setminus \Phi_I^+$ such that $y' := \text{Ad}(u^{-1})(y) \in \mathfrak{u}_{P,K}^-$ fails to be injective on \mathfrak{m}_r for any $y \in \mathfrak{g}_{(-\beta),K} \setminus \{0\}$ (where $\mathfrak{g}_{(-\beta)} := \mathfrak{g}_{-\beta} \oplus \mathfrak{g}_{-2\beta}$). Take any lift $\beta' \in \Phi'^+ \setminus \Phi_{I'}^+$ of β and $y \in \mathfrak{g}'_{-\beta',K} \setminus \{0\}$. Writing $y' = \sum_{\gamma' \in \Phi'^+ \setminus \Phi_{I'}^+} y_{\gamma'}$ with $y_{\gamma'} \in \mathfrak{g}'_{-\gamma'}$ the argument in [OS14b, Theorem 4.7] shows that $\gamma'^+ \in \Phi_{J'}^+$ for the minimal element γ'^+ of $\{\gamma' : y_{\gamma'} \neq 0\}$ for a certain lexicographic order on $\mathbb{Z}_{\geq 0}\Delta'$. (By [OS14b, Corollary 5.5] any nonzero element of $\mathfrak{u}_{P,K}^-$ acts injectively on M .) As $y' = \text{Ad}(u^{-1})(y)$ with $u \in U_P^-$ we have $y_{\beta'} = y \neq 0$, and $y_{\gamma'} \neq 0$ implies $\gamma' \in \beta' + \mathbb{Z}_{\geq 0}(\Delta' \setminus I')$. By minimality of γ'^+ we deduce $\gamma'^+ = \beta'$, so $\beta' \in \Phi_{J'}^+$. Applying the same argument to the lift $\gamma(\beta')$ of β for $\gamma \in \Gamma$, we deduce that $\beta' \in \bigcap_{\gamma \in \Gamma} \gamma^{-1}(\Phi_{J'}^+) = \Phi_{I'}^+$, contradiction. Hence $w \in W_I$ and so we may assume that $g_1 = 1$.

By Lemma A.10 we have $\mathfrak{m}_r = U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$. By Corollary A.7 we have

$$\mathfrak{m}'_r := U_r(\mathfrak{g}') \otimes_{U(\mathfrak{g}')} M = U_r(\mathfrak{g}') \otimes_{U_r(\mathfrak{g})} \mathfrak{m}_r,$$

and moreover $\mathfrak{m}_r \cong \delta_u \star \mathfrak{m}_r$ as $U_r(\mathfrak{g})$ -modules implies $\mathfrak{m}'_r \cong \delta_u \star \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules (as $u \in U_{P,0}^- \subset G_0$). Note that $\mathfrak{m}'_r \neq 0$ because the map $M \rightarrow \mathfrak{m}'_r$ is injective by the first part of the proof (applied to \underline{G}').

Recall that $u \in U_{P,0}^- \subset U_{P',0}^-$. Write $u = u_1 u_2$ with $u_1 \in U_{Q',0}^-$, $u_2 \in U_{P',0}^- \cap L_{Q'}$. We now show that $\delta_{u_2} \star \mathfrak{m}'_r \cong \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules, which implies that $\mathfrak{m}'_r \cong \delta_{u_1} \star \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules. Exactly as in Step 2 of [OS14b, Theorem 4.7] (as $M \in \mathcal{O}^{\mathfrak{q}'}$) we can integrate the locally finite action of $\mathfrak{u}_{P'}^- \cap \mathfrak{l}_{Q'}$ on M to a locally finite locally analytic action of the unipotent group $U_{P'}^- \cap L_{Q'}$. Moreover, $u \circ X \circ u^{-1} = \text{Ad}(u)(X)$ on M for all $u \in U_{P'}^- \cap L_{Q'}$, $X \in \mathfrak{g}'$. (Use, for example, that $\log(\text{Ad}(u)) = \text{ad}(\log(u)) \in \text{GL}(\mathfrak{g}')$ for all $u \in U_{P'}^- \cap L_{Q'}$.) Then we have an isomorphism $\delta_{u_2} \star M \xrightarrow{\sim} M$, $x \mapsto u_2 \cdot x$ as $U(\mathfrak{g}')$ -modules, which extends to an isomorphism $\delta_{u_2} \star \mathfrak{m}'_r \xrightarrow{\sim} \mathfrak{m}'_r$ as the map $U(\mathfrak{g}') \rightarrow U_r(\mathfrak{g}')$ is $U_{P',0}^- \cap L_{Q'}$ -equivariant.

We now have $\mathfrak{m}'_r \cong \delta_{u_1} \star \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules. Applying [OS14b, Theorem 4.7] for the split group \underline{G}' (note that since we are now reduced to $g_1 = 1$ and $g_2 = u_1 \in U_{Q',0}^-$ the proof there only uses that Q' is maximal for $M \in \mathcal{O}^{\mathfrak{q}'}$ and that $\mathfrak{m}'_r = U_r(\mathfrak{g}') \otimes_{U(\mathfrak{g}')} M \neq 0$, i.e. the Q' -action on M is not used!) we deduce that $u_1 \in H'^m Q'_0$ and hence $g_1^{-1} g_2 = u_1 \in H'^m Q'_0 \cap G$.

We next show that $g_1^{-1} g_2 \in H'^m Q'_0 \cap G = H'^m P'_0 \cap G$. Consider the reduction map $\pi: \underline{G}'_{x'}(\mathcal{O}_{L'}) \rightarrow \underline{G}'_{x'}(\mathcal{O}_{L'}/p^{m+m_0})$, which has kernel H'^m by Lemma A.6. For each subset $K' \subset \Delta'$ we get a parabolic subgroup $\underline{P}'_{K',x'}$ of \underline{G}' that contains \underline{T}' , and its schematic closure of $\underline{P}'_{K',x'}$ in $\underline{G}'_{x'}$ is a smooth $\mathcal{O}_{L'}$ -group scheme $\underline{P}'_{K',x'}$ with generic fiber $\underline{P}'_{K',x'}$ [KP22, §2.9]. We note that $\underline{P}'_{K'_1,x'} \times_{\underline{G}'_{x'}} \underline{P}'_{K'_2,x'} = \underline{P}'_{K'_1 \cap K'_2,x'}$ for $K'_1, K'_2 \subset \Delta'$. (By [KP22, Proposition 2.9.1(4), Proposition 2.9.2] the left-hand side is $\mathcal{O}_{L'}$ -smooth, hence $\mathcal{O}_{L'}$ -flat, and we can verify the equality on the geometric generic fiber, where it is clear.) For any $\gamma \in \Gamma$ we get a morphism of buildings $\gamma: \mathcal{B}_{L'}(\underline{G}') \rightarrow \mathcal{B}_{L'}(\underline{G}')$, commuting with the embedding $\mathcal{B}_L(\underline{G}) \hookrightarrow \mathcal{B}_{L'}(\underline{G}')$, hence stabilizing x' . Therefore the γ -linear isomorphism $\underline{G}' \rightarrow \underline{G}'$ over L' extends to a

γ -linear isomorphism $\underline{G}'_{x'} \rightarrow \underline{G}'_{x'}$ over $\mathcal{O}_{L'}$ [KP22, Corollary 2.8.10], and it identifies $\underline{Q}' = \underline{P}'_{J'}$ with $\underline{P}'_{\gamma(J')}$. It follows that $\pi(g_1^{-1}g_2) \in \bigcap_{\gamma \in \Gamma} \underline{P}'_{\gamma(J'), x'}(\mathcal{O}_{L'}/p^{m+m_0}) = \underline{P}'_{x'}(\mathcal{O}_{L'}/p^{m+m_0})$ (where the equality holds by the fiber product computation above, and recall $\underline{P}' = \underline{P}'_{J'}$), so $g_1^{-1}g_2 \in H^m P'_0 \cap G$.

We now know that $g_1^{-1}g_2 \in H^m P'_0 \cap G$. As $\underline{U}_{\underline{P}} \times \underline{P} \rightarrow \underline{G}$ is an open immersion of schemes over L , we finally deduce that $g_1^{-1}g_2 \in (H'^{-,m} \cap U_{\underline{P}})(P'_0 \cap P) = H'^{-,m} P_0$. For the equality $H'^{-,m} \cap U_{\underline{P}} = H'^{-,m}$ we used that the logarithm map is bijective (even a locally analytic isomorphism) between $U_{\underline{P}}$ and $\mathfrak{u}_{\underline{P}}$ (as $\underline{U}_{\underline{P}}$ is unipotent) and that $(p^{m+m_0} \mathfrak{g}'_0 \cap \mathfrak{u}_{\underline{P}}) \cap \mathfrak{u}_{\underline{P}} = p^{m+m_0} \mathfrak{g}_0 \cap \mathfrak{u}_{\underline{P}}$. \square

Remark A.14. The result is true even if we only assume that $M \in \mathcal{O}^{\mathfrak{p}}$ (satisfying (i) and (ii)) and if we define $\mathfrak{m}_r := U_r(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M$. The simplicity proof goes through as before. Now suppose that $\delta_{g_1} \star \mathfrak{m}_r \cong \delta_{g_2} \star \mathfrak{m}_r$ as $U_r(\mathfrak{g})$ -modules for some $g_i \in G_0$. By Corollary A.7 we have $\delta_{g_1} \star \mathfrak{m}'_r \cong \delta_{g_2} \star \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules, where $\mathfrak{m}'_r := U_r(\mathfrak{g}') \otimes_{U(\mathfrak{g}')} M = U_r(\mathfrak{g}') \otimes_{U_r(\mathfrak{g})} \mathfrak{m}_r$. If $\underline{G}^{\text{der}}$ is simply connected, then $M \in \mathcal{O}^{\mathfrak{q}'}$ lifts to an object of $\mathcal{O}^{\mathfrak{Q}'}$, after perhaps replacing C by a finite extension, by Lemma 2.22. By [OS14b, Theorem 4.7] for the split group \underline{G}' we deduce $g_1^{-1}g_2 \in H^m Q'_0 \cap G$, and the final part of the above proof shows that $H^m Q'_0 \cap G = H'^{-,m} P_0$. For general \underline{G} , let $\pi : \tilde{G}' \rightarrow \underline{G}'$ be a z -extension, so $(\tilde{G}')^{\text{der}}$ is simply connected and $\pi : \tilde{G}' \rightarrow \underline{G}'$ surjects on L' -points. We may and will suppose that \tilde{G}' and the torus $\ker(\pi)$ are split as well. By [KP18, Proposition 1.1.4], π extends to a surjective map $\tilde{G}'_{x'} \rightarrow \underline{G}'_{x'}$ of (connected reductive) parahoric group schemes over $\mathcal{O}_{L'}$ with kernel a split torus. In particular, by checking on special fibers, the map on $\mathcal{O}_{L'}$ -Lie algebras $\tilde{\mathfrak{g}}'_0 \rightarrow \mathfrak{g}'_0$ is surjective, where $\tilde{\mathfrak{g}}'_0 := \text{Lie } \tilde{G}'_{x'}$. (Lie algebras of smooth group schemes commute with base change.) Also, by taking fppf-cohomology we obtain $\pi(\tilde{G}'_0) = G'_0$. Following our construction above, we have $\tilde{H}' := \text{BCH}(p^{m_0} \tilde{\mathfrak{g}}'_0) \triangleleft \tilde{G}'_0$ and $\pi(\tilde{H}') = H'$. Letting \tilde{Q}' be the pre-image of \underline{Q}' under π and $\tilde{Q}'_0 := \tilde{Q}' \cap \tilde{G}'_0$ we have $\pi(\tilde{Q}'_0) = Q'_0$. As the map $\tilde{\mathfrak{g}}'_0 \rightarrow \mathfrak{g}'_0$ splits as \mathbb{Z}_p -modules, there exists a minimal set \tilde{x}'_i ($1 \leq i \leq d'$) of topological generators of \tilde{H}' such that $x'_i := \pi(\tilde{x}'_i)$ ($1 \leq i \leq d$) is a minimal set of topological generators of H' and $\pi(\tilde{x}'_i) = 1$ for $d < i \leq d'$ (for some $d' \geq d \geq 0$). The construction in [OS10, §2.2.6] then shows that the natural map $D^{(L)}(\tilde{H}') \rightarrow D^{(L)}(H')$ is continuous with respect to $|\cdot|_r$ -norms, so induces a map $D_r^{(L)}(\tilde{H}') \rightarrow D_r^{(L)}(H')$ (in fact, both maps are strict surjections). By taking the closures of enveloping algebras we obtain a commutative square

$$\begin{array}{ccc} U(\tilde{\mathfrak{g}}') & \longrightarrow & U(\mathfrak{g}') \\ \downarrow & & \downarrow \\ U_r(\tilde{\mathfrak{g}}', \tilde{H}') & \dashrightarrow & U_r(\mathfrak{g}', H') \end{array}$$

where the bottom map is a morphism of Banach algebras and $\tilde{\mathfrak{g}}' := \text{Lie } \tilde{G}'$. By density of the enveloping algebras, all maps are equivariant for the adjoint action of $\tilde{G}'_0 \rightarrow G'_0$. Pick $\tilde{g}'_i \in \tilde{G}'_0$ for $i = 1, 2$ such that $\pi(\tilde{g}'_i) = g_i$ in G'_0 . From $\delta_{g_1} \star \mathfrak{m}'_r \cong \delta_{g_2} \star \mathfrak{m}'_r$ as $U_r(\mathfrak{g}')$ -modules we deduce that $\delta_{\tilde{g}'_1} \star \mathfrak{m}'_r \cong \delta_{\tilde{g}'_2} \star \mathfrak{m}'_r$ as $U_r(\tilde{\mathfrak{g}}')$ -modules. Hence, as $(\tilde{G}')^{\text{der}}$ is simply connected, we deduce $(\tilde{g}'_1)^{-1} \tilde{g}'_2 \in \tilde{H}^m \tilde{Q}'_0$. Applying π we obtain $g_1^{-1}g_2 \in H^m Q'_0 \cap G$ and we conclude as before.

We can now complete the proof of Theorem A.1.

Proof of Theorem A.1(iii). In the proof of [OS14b, Proposition 3.7] an Iwahori–Bruhat decomposition is used. Here we temporarily forget our choice of G_0 and P_0

above and can use any special point x (e.g. the one above). Let \underline{G}_x over \mathcal{O}_L denote the (connected smooth) parahoric group scheme of [BT84], with generic fiber \underline{G} . Let I_0 be the Iwahori that is the inverse image under $G_0 := \underline{G}_x(\mathcal{O}_L) \rightarrow \underline{G}_x^{\text{red}}(k_L)$ of \underline{B}_x , where \underline{B}_x is the minimal parabolic defined by Φ^+ in the reductive quotient $\underline{G}_x^{\text{red}}$ of $\underline{G}_x \times_{\mathcal{O}_L} k_L$. (By [BT84], \underline{S} extends to \mathcal{O}_L , $\underline{S}_x \times k_L$ is a maximal split torus of $\underline{G}_x^{\text{red}}$ and the root system of $(\underline{G}_x^{\text{red}}, \underline{S}_x \times k_L)$ is naturally a subset of that of $(\underline{G}, \underline{S})$.) Then we get an Iwahori–Bruhat decomposition $G_0 = \coprod_{w \in W/W_I} I_0 \dot{w}(P \cap G_0)$ using the argument of [AHHV17, §IV.5].

For the proof of [OS14b, Theorem 4.5] we define $G_0, P_0, H = H^-H^+, H^m = H^{-,m}H^{+,m}$ as in the proof of Theorem A.13. The proof of [OS14b, Theorem 4.5] (and [OS15, Theorem 5.8]) then proceeds just as before, except that we apply Theorem A.13 instead of [OS14b, Theorem 4.7]. \square

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