TALK 2

Serre-type Conj., mod p coho., and rep. theory of $GL_n(\mathbb{Q}_p)$

(Florian Herzig)

① Compact unitary gps.

$E/\mathbb{Q}$ imag. quad., split at p, $\quad Gal(E/\mathbb{Q}) = \langle c \rangle$

$\langle , \rangle$: pos. def. hermitian form on E^n.

$G/\mathbb{Q}$: unitary gp. of $\langle , \rangle$:

$$\forall\ \mathbb{Q}\text{-alg. } A, \quad G(A) = \{ g \in GL_n(A \otimes_{\mathbb{Q}} E) : g \text{ preserves } \langle , \rangle \}.$$

Then $G \times_{\mathbb{Q}} \mathbb{R} \cong U(n)$, and $\underbrace{G \times_{\mathbb{Q}} \mathbb{Q}_\ell \cong GL_n}_{\text{fix. isos.} \leftrightarrow \text{fix } \lambda | \ell}$ $\forall \ell$ split in E. (in particular, $\ell = p$)

Let $U \subset G(\mathbb{A})$ be cpt. open subgp.

$$U = U_\infty \times U_p \times U^{\infty,p} \quad \text{with} \quad \begin{aligned} U_\infty &= G(\mathbb{R}) \quad (\text{cpt!}) \\ U_p &= GL_n(\mathbb{Z}_p) \quad (\text{"level prime to } p\text{"}) \\ U^{\infty,p} & \text{ suff. small} \end{aligned}$$

Have cov. space:

$$\begin{tikzcd} X_U^p := G(\mathbb{Q}) \backslash G(\mathbb{A}) / U^p \arrow[d] & \circlearrowleft G(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p). \\ X_U := G(\mathbb{Q}) \backslash G(\mathbb{A}) / U & (0\text{-dim}^l \text{ loc. symm. space}) \end{tikzcd}$$

For W a rep. of U_p, have loc. const. sheaf Z_W on X_U.

$$M(U, W) := H^0(X_U, Z_W) = \mathrm{Hom}^{cont}_{U_p}(X_U^p, W) \quad \underline{f.d.}$$

"alg. modular forms of wt. W" (Gross)

$$\circlearrowleft$$

$\mathbb{T} := \mathbb{Z}[T_{\ell,1}, \ldots, T_{\ell,n} :$
ℓ split in E, $\ell \neq p$,
$U_\ell = GL_n(\mathbb{Z}_\ell)]$

Hecke: Write $\mathrm{GL}_n(\mathbb{Z}_\ell)\begin{pmatrix} \overbrace{\ell \ddots \ell}^{i} & \\ & 1 \ddots 1 \end{pmatrix}\mathrm{GL}_n(\mathbb{Z}_\ell) = \coprod_{\text{fin.}} \gamma_\alpha \mathrm{GL}_n(\mathbb{Z}_\ell)$.

Then $(T_{\ell,i} f)(x) = \sum_\alpha f(x\gamma_\alpha)$. (here $f : X_U^p \to W$)

Two choices for W:

① W a rep. of $G \times_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{GL}_n$:

a $\mathbb{T}$-evec. $f \in \mathcal{M}(U, W)$ has an attached

Galois rep. $\rho_f : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ s.t. $\rho_f \circ c \cong \rho_f^\vee$

→ known for $n=3$: Rogawski, etc.
and general n (under certain local conditions)
Kottwitz, Clozel

② $W = F(\lambda)$ a Serre weight:

$$U_p = \mathrm{GL}_n(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_p) \curvearrowright F(\lambda).$$

Using $\mathcal{M}(U, F(\lambda)) \subset \mathcal{M}(U, W(\lambda))$ and lifting to char. 0

(D-S): $\mathbb{T}$-evec. $f \in \mathcal{M}(U, F(\lambda)) \rightsquigarrow \rho_f : G_E \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$,
$\rho_f \circ c \cong \rho_f^\vee$.

→ known: see above remarks

Rk: Frob_ℓ for good ℓ are dense in G_E!

- Serre-type Conj.:

$\rho : G_E \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ irr., $\rho \circ c \cong \rho^\vee$

$\rightsquigarrow W(\rho) = \{$ Serre wts. $F \mid \rho$ attached to $\mathbb{T}$-evec. in $\mathcal{M}(U, F)$, some U as above $\}$

Same $W^?(\rho)$ as in prev. talk.

Conj.: $W(\rho) = W^?(\rho)$ if ρ occurs in cohomology

i.e. $H^0(X_U^p, \overline{\mathbb{F}}_p)[m_\rho] \neq 0$ for some U^p

[there could be local obstructions for $\ell \neq p$]

Different viewpoint:

$$\mathcal{M}(U,F) = H^0(X_U, \mathcal{L}_F) \cong \mathrm{Hom}_{U_p}\big(F^\vee,\ \underset{\substack{\circlearrowleft \\ G(\mathbb{Q}_p)}}{H^0(X_U^p, \overline{\mathbb{F}}_p)}\big) \qquad (*)$$

Note: $H^0(X_U^p, \overline{\mathbb{F}}_p) = \varinjlim_{U_p' \subset U_p \text{ open}} H^0(X_{U^p U_p'}, \overline{\mathbb{F}}_p)$

as in Emerton's talk.

Let $\mathfrak{m}_\rho \triangleleft \mathbb{T}$ be max. ideal assoc. to ρ.

$$W(\rho) = \{ F : F^\vee \underset{U_p\text{-lin.}}{\hookrightarrow} H^0(X_U^p, \overline{\mathbb{F}}_p)[\mathfrak{m}_\rho], \text{ some } U \text{ as above} \}.$$

② Hecke action at p

From $(*)$,

$$\mathcal{M}(U,F) \cong \mathrm{Hom}_{G(\mathbb{Q}_p)}\big(\underbrace{\overline{\mathbb{F}}_p[G(\mathbb{Q}_p)] \otimes_{\overline{\mathbb{F}}_p[U_p]} F^\vee},\ H^0(X_U^p, \overline{\mathbb{F}}_p)\big)$$

$\cong \{$functions $\varphi : G(\mathbb{Q}_p) \to F^\vee$

- $\varphi(ug) = u\varphi(g) \quad \forall u \in U_p,\ g \in G(\mathbb{Q}_p)$
- φ loc. const., cpt. supp. $\}$.

U_p cpt. open

$=: \mathrm{ind}_{U_p}^{G(\mathbb{Q}_p)}(F^\vee)$ (compact induction)

$\therefore$ The $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_{U_p}(F^\vee) := \mathrm{End}_{G(\mathbb{Q}_p)}(\mathrm{ind}_{U_p}^{G(\mathbb{Q}_p)} F^\vee)$

acts on $\mathcal{M}(U,F)$, commuting with $\mathbb{T}$ action.

Notation: $K := U_p$, $G := G(\mathbb{Q}_p)$.

Viewpoints:

(i) Biinvariant functions:

$$\mathcal{H}_K(F) = \mathrm{Hom}_G(\mathrm{ind}_K^G F, \mathrm{ind}_K^G F) = \mathrm{Hom}_K(F, \underbrace{\mathrm{ind}_K^G F}_{\text{maps } G \to F})$$

$$\cong \left\{ f: G \to \mathrm{End}\, F : \begin{array}{l} \cdot\ f(k_1 g k_2) = k_1 f(g) k_2 \quad \forall\, k_i \in K,\ g \in G \\ \cdot\ \text{loc. const., cpt. supp.} \end{array} \right.$$

under convolution:

$$(f_1 * f_2)(g) = \sum_{\gamma \in G/K} f_1(g\gamma) f_2(\gamma^{-1})$$

Note: $\mathcal{H}_K(\mathbb{1}) \cong \overline{\mathbb{F}}_p[K \backslash G / K]$

(ii) Yoneda:

$\mathcal{H}_K(F)^{op}$ consists of endos. of functor

$$\begin{array}{c} \mathrm{Hom}_G(\mathrm{ind}_K^G F, -) : G\text{-mod} \longrightarrow \underline{\mathrm{Set}} \\ \| \\ \mathrm{Hom}_K(F, -) \end{array} \qquad [F = \mathbb{1} : \pi \longmapsto \pi^K]$$

Structure:

Let $\bar{N} := \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix}$

Basic fact: $F^{\bar{N}(\mathbb{F}_p)}$ is one dim^l. (lowest wt. space)

$\circlearrowright$

$T(\mathbb{F}_p)$ acts by char. χ_F.

Let $T^+ := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in T : \ \mathrm{ord}(t_1) \geq \cdots \geq \mathrm{ord}(t_n) \right\}$.

Thm. (H.)

There is an inj. homo. of $\overline{\mathbb{F}}_p$-algebras

$$\mathcal{H}_K(F) \xhookrightarrow{\text{"Satake"}} \mathcal{H}_{T(\mathbb{Z}_p)}(\chi_F)$$

$$f \longmapsto \left(t \longmapsto \left(\sum_{\bar{n} \in \bar{N}/\bar{N}(\mathbb{Z}_p)} f(t\bar{n}) \right) \Bigg|_{F^{\bar{N}(\mathbb{F}_p)}} \right)$$

with image

$\mathcal{H}^+_{T(\mathbb{Z}_p)}(\chi_F) := \{ \varphi : T \longrightarrow \overline{\mathbb{F}}_p :$

- $\varphi(t_0 t) = \chi_F(t_0)\, \varphi(t) \quad \forall\, t_0 \in T(\mathbb{Z}_p), t \in$
- loc. const., cpt. supp.
- $\mathrm{supp}\, \varphi \subset T^+ \}$.

$\cong \overline{\mathbb{F}}_p[Y(T)_+]$

↖ choice of uniformiser

Cor.: $\mathcal{H}_K(F)$ comm.

Rk: Same formula as in classical case ($/\mathbb{C}$), but drop modulus char. (power of p).

Classically, image $\cong \mathbb{C}[Y(T)]^W$. (here toric)

Proof outline:

Cartan dec.: $$G = \coprod_{\mu \in Y(T)_+} K\mu(p)K$$

Let $\mu \in Y(T)_+$, $t = \mu(p)$.

red : $K = GL_n(\mathbb{Z}_p) \twoheadrightarrow GL_n(\mathbb{F}_p)$

Lemma: red$(K \cap K^t) = P(\mathbb{F}_p)$, where $P = MU$ is parab. associated to μ.

red$({}^tK \cap K) = \bar{P}(\mathbb{F}_p)$, opp. parab.

The map $F^{\bar{U}(\mathbb{F}_p)} \hookrightarrow F \twoheadrightarrow F_{U(\mathbb{F}_p)}$ is an iso. of irr. $M(\mathbb{F}_p)$-reps.

Find: $\exists!\, T_\mu \in \mathcal{H}_K(F)$ s.t.

(a) supp $T_\mu = K\mu(p)K$

(b) $T_\mu(\mu(p)) : F \twoheadrightarrow F_{U(\mathbb{F}_p)} \xleftarrow{\sim} F^{\bar{U}(\mathbb{F}_p)} \hookrightarrow F$

Also: (a) determines T_μ up to scalar.

Then proceed as in classical case. □

Prop. (H.)
Suppose $F = F(\lambda)$, $\lambda \in X_1(T)$ with $\langle \lambda, \alpha_i^\vee \rangle \neq 0\ \forall i$.
Then $T_\mu T_\nu = T_{\mu+\nu}\ \ \forall \mu, \nu$.

$G = GL_n$: $$\mu_i(t) = \begin{pmatrix} \underbrace{t \ \ddots \ t}_{i} & \\ & 1 \ \ddots \ 1 \end{pmatrix}$$ in $Y(T)_+$ ("basis")

$T_i := T_{\mu_i}$. Then $\mathcal{H}_K(F) \cong \mathbb{F}_p[T_1, \dots, T_{n-1}, T_n^{\pm 1}]$.

n = 2 : Barthel-Livné

Comparison with classical Hecke operators

$W(\lambda)_{\overline{\mathbb{Z}}_p} \subset W(\lambda)_{\overline{\mathbb{Q}}_p}$ -- $U_p = GL_n(\mathbb{Z}_p)$-stable lattice.

$\rightsquigarrow \mathcal{M}(U, W(\lambda)_{\overline{\mathbb{Z}}_p}) \subset \mathcal{M}(U, W(\lambda)_{\overline{\mathbb{Q}}_p})$ -- $\mathbb{T}$-stable lattice with reduction $\mathcal{M}(U, W(\lambda)_{\overline{\mathbb{F}}_p})$.

$$\begin{tikzcd}\mathcal{M}(U, W(\lambda)_{\overline{\mathbb{Q}}_p}) \arrow[loop below, "T_\mu^{cl}"]\end{tikzcd}$$

$$(T_\mu^{cl} f)(x) = \sum_{g \in K\mu(p)K/K} g\, f(xg)$$

Note: the T_μ^{cl} ($\mu \in Y(T)_+$) and $\mathbb{T}$ all commute.

Let $w_0 \in W$ be the longest elt. (rep^d by $\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in N(T)$).

Prop. (EGH): $p^{-\langle w_0 \lambda, \mu \rangle} T_\mu^{cl}$ preserves $\mathcal{M}(U, W(\lambda)_{\overline{\mathbb{Z}}_p})$ and induces T_μ on $\mathcal{M}(U, F(\lambda)) \subset \mathcal{M}(U, W(\lambda)_{\overline{\mathbb{F}}_p})$.

(Point: consider how $\mu(p)$ acts on wt. spaces of $W(\lambda)_{\overline{\mathbb{Q}}_p}$).

③ Application to Serre-type Conjectures

$$\rho : G_E \longrightarrow GL_n(\overline{\mathbb{F}}_p) \text{ irr.}, \quad \rho \circ c \cong \rho^\vee$$

Suppose $F = F(\overbrace{a_1, \dots, a_n}^{\lambda}) \in W(\rho)$

$$\therefore \quad \mathcal{M}(U, F)[m_\rho] \neq 0 \qquad (\text{some } U)$$
$$\cup$$
$$\mathcal{H}_K(F^\vee)$$

[f.d.]

Prop. (EGH) Suppose $n = 3$ (to be safe).
If $\exists\ 0 \neq f \in \mathcal{M}(U, F)[m_\rho]$ s.t. $T_i f = \xi_i f$, $\xi_i \neq 0$ $(1 \leq i \leq n)$

then
$$\rho|_{I_p} \sim \begin{pmatrix} \omega^{a_1+n-1} & & * \\ & \ddots \ \omega^{a_{n-1}+1} & \\ & & \omega^{a_n} \end{pmatrix}.$$

Proof: By prev. prop. and Deligne-Serre lifting lemma, there is an evec. $\tilde{f} \in \mathcal{M}(U, W(a_1, \dots, a_n)_{\overline{\mathbb{Q}}_p})$ for $\mathbb{T}$ and the $p^{-\langle w_0\lambda, \mu\rangle} T_\mu^{cl}$ $(\mu \in Y(T)_+)$ s.t. the evals. lift those of $\mathbb{T}$ and T_μ on f.

$\therefore$ T_i^{cl}-eval. on $\tilde{f}$ has valuation $\langle w_0\lambda, \mu_i\rangle = a_n + \dots + a_{n+1-}$

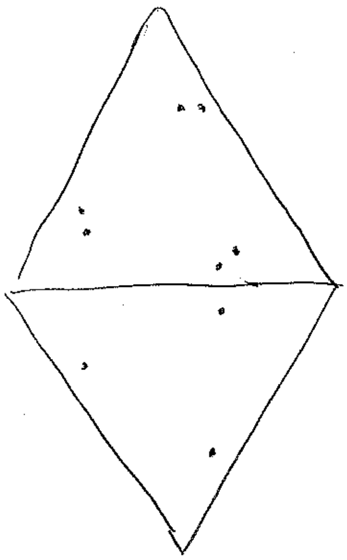
The associated Galois rep. $\rho_{\tilde{f}} : G_E \longrightarrow GL_n(\overline{\mathbb{Q}}_p)$ lifts ρ, is cryst. at p, has HT wts. $(a_1+n-1, \dots, a_n)$ and $\varphi \curvearrowright D_{cris}(\rho_{\tilde{f}}|_{D_p})$ has slopes $a_1+n-1, \dots, a_n$.

Thus $\rho_{\tilde{f}}|_{D_p}$ is ordinary and so $\rho_{\tilde{f}}|_{I_p} \sim \begin{pmatrix} \chi^{a_1+n-1} & & * \\ & \ddots & \\ & & \chi^{a_n} \end{pmatrix}$.

(χ = p-adic cyclo.) □

Rk: Similar, but weaker, result if only some of the T_i have non-zero eval.

E.g.: $n = 3$, tame $\rho|_{I_p}$ of niveau 1. + generic



The 6 obv. wts. should be ordinary [can guess now what they are...]

The 3 shadows have to be supersing. ($T_1 = T_2 = 0$)