SMOOTH REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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1. Motivation

Smooth representations of p-adic groups arise in number theory mainly through the study of automorphic representations, and thus in the end, for example, from modular forms.

We saw in the first lecture by Matt Emerton that a modular form, thought of as function on the set of lattices with level N structure, we obtain a function in $C^\infty(GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Z}/N), \mathbb{C})$ satisfying certain differential equations (coming from holomorphicity and the weight of the modular form) and growth conditions (related to the cusps). As was suggested, this space has an adelic interpretation: by the strong approximation theorem, $GL_2(\mathbb{A}_\mathbb{Q}) = GL_2(\mathbb{Q})GL_2(\widehat{\mathbb{Z}})GL_2(\mathbb{R})$, and a simple argument shows that

$$C^\infty(GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R}) \times GL_2(\mathbb{Z}/N), \mathbb{C}) \cong C^\infty(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q})/U(N), \mathbb{C}),$$

where $U(N) = \ker(GL_2(\widehat{\mathbb{Z}}) \twoheadrightarrow GL_2(\mathbb{Z}/N))$. Taking the union of these spaces over all $N \geq 1$ (still with holomorphicity and growth conditions imposed), we get a big "automorphic" space $\mathcal{A}^\infty \subset C^\infty(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q}), \mathbb{C})$ which has a natural action of $GL_2(\mathbb{A}_\mathbb{Q}^\infty)$ by right translations. This action is "smooth": by definition, every element of $\mathcal{A}^\infty$ is fixed by some open subgroup, namely $U(N)$ for some N.

Moreover if we start with a cuspidal newform f, we can get a subrepresentation $\pi^\infty \subset \mathcal{A}^\infty$ by restricting ourselves to all oldforms of f in each level. (This is a little vague, as we are working with functions living on several copies of the classical modular curve $X(N)$, but can be made precise.) This is in fact irreducible and has a restricted tensor product decomposition $\pi^\infty \cong \bigotimes'_p \pi_p$, where each π_p is a representation of $GL_2(\mathbb{Q}_p)$. Since the action of $GL_2(\mathbb{A}_\mathbb{Q}^\infty)$ on π^∞ is smooth, so is the action of $GL_2(\mathbb{Q}_p)$ on π_p: each element of π_p is fixed by an open subgroup, namely $\ker(GL_2(\mathbb{Z}_p) \twoheadrightarrow GL_2(\mathbb{Z}/p^m))$ for some $m \geq 1$.

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For representation-theorists there is another (related) way in which smooth representations arise. Suppose we want to study representations of a non-compact topological group like $G = GL_2(\mathbb{Q}_p)$. It is natural to consider unitary G-actions on a Hilbert space $\mathcal{H}$ such that the action map $G \times \mathcal{H} \to \mathcal{H}$ is continuous. Note that the subgroup $K = GL_2(\mathbb{Z}_p)$ is compact, so that $\mathcal{H}|_K$ is a Hilbert space direct sum of irreducible (finite-dimensional) K-representations. It follows that the space of K-finite vectors,

$$\begin{aligned} \mathcal{H}^{K\text{-fin}} &:= \{x \in \mathcal{H} : \mathbb{C}Kx \text{ is finite dimensional}\} \\ &= \bigcup_n \mathcal{H}^{1+p^m M_2(\mathbb{Z}_p)} \end{aligned}$$

is dense in $\mathcal{H}$ (as $1 + p^m M_2(\mathbb{Z}_p)$ is a system of neighbourhoods of the identity in G and $GL_d(\mathbb{C})$ has "no small subgroups," any irreducible unitary K-representation is K-finite). It is a G-stable subspace (which is in fact irreducible if and only if $\mathcal{H}$ is topologically irreducible) and the G-action on it is smooth: every element is fixed by an open subgroup.

2. Basic definitions

Suppose F is a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$, uniformiser ϖ, and residue field k. Let $\mathbf{G}$ be a connected reductive group over F, which we will assume for simplicity to be *split* (all results, appropriately interpreted, still make sense in general). Let $G := \mathbf{G}(F)$. When we talk about Borel subgroups, parabolic subgroups, tori, etc. of G, we will mean the F-points of the the corresponding subgroups of $\mathbf{G}$.

We give G the coarsest topology so that all functions $G \to F$ in the coordinate ring $F[\mathbf{G}]$ become continuous (where F is given its natural topology induced by its p-adic valuation). Then G is a p-adic Lie group over F (also known as locally analytic group), so it is totally disconnected, and the compact open subgroups in G are a neighbourhood basis at the identity. (See Serre's book on Lie groups for the elementary proof.)

For example, G could be $GL_n(F)$, $GSp_4(F)$, etc. If $G = GL_n(F)$, it has the subspace topology in $M_n(F) \cong F^{n^2}$, and the congruence subgroups $K_m := 1 + \varpi^m M_n(\mathcal{O}_F)$ $(m \geq 1)$ are a neighbourhood basis at the identity.

Exercise 1. *Show that $K := GL_n(\mathcal{O}_F)$ is a maximal compact subgroup in $GL_n(F)$ and that every other maximal compact subgroup is conjugate to K. (Hint: Show that every compact subgroup stabilises a lattice in F^n.)*

Definition 2.1. *A representation of G on a complex vector space V is called* smooth *if one of the following equivalent conditions hold:*

(i) $\forall v \in V, \quad \mathrm{Stab}_G(v)$ *is open,*
(ii) $V = \bigcup_U V^U$, *where U runs over all compact open subgroup U,*
(iii) *The action map $G \times V \to V$ is continuous, if V is given the* discrete *topology.*

(This definition makes sense equally well over other base fields. Smooth representations in characteristic p play a role in the hypothetical mod p Langlands correspondence.)

A map between smooth representations V and W is simply a $\mathbb{C}G$-linear map $V \to W$. With this definition, we get an abelian category of smooth representations of G (with usual kernels and cokernels).

Note that a smooth character $F^\times \to \mathbb{C}^\times$ is a homomorphism which is trivial on $K_m = 1 + \varpi^m \mathcal{O}_F$ for some $m \geq 1$.

Exercise 2. *Any* finite-dimensional *irreducible smooth representation of $GL_n(F)$ is of the form $\chi \circ \det$, where χ is a smooth character of $F^\times$. (Hint: Show that such a representation has to be trivial on the upper triangular unipotent matrices, and note that their conjugates generate $SL_n(F)$.)*

If V is a smooth representation, the linear dual $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ (with usual G-action $(gf)(v) = f(g^{-1}v)$) may not be smooth. The *contragredient* $\widetilde{V}$ is the maximal smooth subrepresentation: $\widetilde{V} := \bigcup (V^*)^U$, where U runs over all compact open subgroups.

Definition 2.2. *A smooth representation is said to be* admissible *if* $\dim_{\mathbb{C}} V^U < \infty$ *for all compact open subgroups $U \leq G$.*

The full subcategory of admissible representations is closed under taking kernels, cokernels, and direct sums, and hence abelian in its own right.

Exercise 3.

(i) *An irreducible smooth representation of a compact subgroup H is finite dimensional.*
(ii) *Fix a compact open subgroup U. Then V is admissible iff $V|_U$ is a direct sum of irreducibles, each occurring with finite multiplicity.*

The admissibility condition lets us bootstrap some results that are true for finite-dimensional representations, as in the following exercise.

Exercise 4.

(i) *If V is smooth and U a compact open subgroup,*
$$\begin{aligned} e_U : V &\twoheadrightarrow V^U \\ v &\mapsto \int_U uv\, du \end{aligned}$$
splits the inclusion $V^U \subset V$ (where du is the left- and right-invariant Haar measure on U with total volume 1). In particular $V^U \cong V_U$ (the U-coinvariants) and $V \mapsto V^U$ is an exact functor.
(ii) *Suppose that V is admissible. Then $\widetilde{V}$ is admissible and the natural map $V \to \widetilde{\widetilde{V}}$ is an isomorphism. Also V is irreducible iff $\widetilde{V}$ is.*

Suppose that $H \leq G$ is a closed subgroup and that V is a smooth representation of H. Imitating the construction of the induced representation for finite groups, we consider

$$\begin{aligned} I &= \{f : G \to V : f(hg) = hf(g) \quad \forall g \in G,\ h \in H\} \\ &= \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, V) \end{aligned}$$

with action $(xf)(g) = f(gx)$. In general this G-action will not be smooth, but G stabilises the smooth subrepresentation $I^\infty := \bigcup_U I^U$ which we will call the *induced representation* $\mathrm{Ind}_H^G V$. Concretely,

$$\begin{aligned} \mathrm{Ind}_H^G V = \{f : G \to V : f(hg) = hf(g)\ \forall g \in G,\ h \in H, \\ \text{and } \exists U \text{ c.o.s. such that } f(gu) = f(g)\ \forall u \in U\}. \end{aligned}$$

Note that any $f \in \mathrm{Ind}_H^G V$ has open and closed support in $H\backslash G$. There is a natural subrepresentation, the *compactly induced representation* $\text{c-Ind}_H^G V$ (also sometimes denotes $\mathrm{ind}_H^G V$):

$$\text{c-Ind}_H^G V = \{f \in \mathrm{Ind}_H^G V : \mathrm{Supp}\, f \text{ is compact in } H\backslash G\}.$$

Remark 2.3. *Note that for a function $f \in I$ it is not enough to be locally constant in order to lie in $\mathrm{Ind}_H^G V$. This is however easily seen to be sufficient if $\mathrm{Supp}\, f$ is compact in $H\backslash G$.*

Proposition 2.4. *(Frobenius reciprocity) If V is a smooth G-representation and W a smooth H-representation, then*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W) \cong \mathrm{Hom}_H(V|_H, W).$$

Proof. This reduces to the usual statement in algebra: as V is smooth, we can replace $\mathrm{Ind}_H^G W$ on the left by $\mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, W)$. □

Exercise 5. *If U is an* open *subgroup, then $W \mapsto \text{c-Ind}_U^G W \cong \mathbb{C}G \otimes_{\mathbb{C}U} W$ is a left adjoint of $V \mapsto V|_U$.*

Exercise 6.

(i) *If $H \leq J \leq G$ are closed subgroups and W a smooth representation of H then*

$$\mathrm{Ind}_J^G(\mathrm{Ind}_H^J W) \cong \mathrm{Ind}_H^G W.$$

The same statement also holds if we take compact induction everywhere.

(ii) *If V is a smooth G-representation and W a smooth H-representation, then*

$$\text{c-Ind}_H^G(V|_H \otimes W) \cong V \otimes \text{c-Ind}_H^G W.$$

The same holds for full inductions if W is finite-dimensional.

(iii) $\widetilde{\text{c-Ind}_H^G V} \cong \mathrm{Ind}_H^G(\widetilde{V} \otimes \delta_H)$. *(See (3.5) for δ_H.)*

Proposition 2.5.

(i) *The functors Ind_H^G and c-Ind_H^G are exact.*

(ii) *If W is admissible and $H\backslash G$ is compact, then* c-$\mathrm{Ind}_H^G W$ *is admissible.*

Proof. It suffices to show that $(\mathrm{Ind}_H^G(-))^U$ is exact for any compact open subgroup U. Choose a set of representatives g for $H\backslash G/U$. Then

$$\begin{aligned}(\mathrm{Ind}_H^G W)^U &\xrightarrow{\sim} \prod_{g\in H\backslash G/U} W^{H\cap {}^gU}\\ f &\mapsto \quad (f(g)).\end{aligned}$$

(The point is that f is determined by its values on the set of representatives, and the condition $f(hgu) = hf(g)$ shows that $f(g)$ has to land in $W^{H\cap {}^gU}$.) Similarly,

$$\begin{aligned}(\text{c-}\mathrm{Ind}_H^G W)^U &\xrightarrow{\sim} \bigoplus_{g\in H\backslash G/U} W^{H\cap {}^gU}\\ f &\mapsto \quad (f(g)).\end{aligned}$$

Then (i) follows since $H\cap {}^gU \leq H$ is compact open and we have shown that invariants by a compact open subgroup is exact (Ex. 4).

Also (ii) is now trivial since $H\backslash G/U$ is finite by compactness of $H\backslash G$. $\square$

3. Parabolic induction and Jacquet functors

Suppose now that P is a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$. Recall that in $GL_n(F)$ these are (up to conjugation) given by

$$M = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}, \quad N = \begin{pmatrix} 1_{d_1} & * & \dots & * \\ & 1_{d_2} & \dots & * \\ & & \ddots & \vdots \\ & & & 1_{d_r} \end{pmatrix},$$

where $A_i \in GL_{d_i}(F)$ and $\sum d_i = n$.

If σ is a smooth representation of M, we can consider it as smooth P-representation by the quotient map $P \twoheadrightarrow M$. The *parabolic induction* of σ to G is $\mathrm{Ind}_P^G \sigma = \text{c-}\mathrm{Ind}_P^G \sigma$. The equality holds since by definition $\mathbf{P}\backslash\mathbf{G}$ is a proper variety so that $P\backslash G$ is compact.

On the other hand if π is a smooth representation of G, we can consider the N-coinvariants, the largest quotient on which N acts trivially:

$$J_N(\pi) = \pi/\pi(N), \text{ where } \pi(N) = \langle nx - x : n \in N, x \in \pi\rangle.$$

This is called the *Jacquet functor* of π (with respect to P). Since $N \lhd P$, $J_N(\pi)$ has a natural action of $M = P/N$ and this is automatically smooth.

Exercise 7.

(i) *Note that as N is a unipotent subgroup, there are compact open subgroups $N_i \leq N$ such that $N = \bigcup N_i$ (think of the case of GL_n!).*

Show that

$$\pi(N) = \{x \in \pi : \exists i, \int_{N_i} nx\, dx = 0\}.$$

(ii) *Use (i) to see that J_N is an exact functor. (It is only unclear whether it preserves injections.)*

Theorem 3.1. *(Jacquet) If π is admissible, so is $J_N(\pi)$.*

Frobenius reciprocity has the following immediate corollary.

Proposition 3.2. *If σ is a smooth M-representation and π a smooth G-representation, then*

$$\operatorname{Hom}_M(J_N(\pi), \sigma) \cong \operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G \sigma).$$

Remark 3.3. *In fact, restricting to the categories of admissible representations, J_N also has a left adjoint, namely $\operatorname{Ind}_{\bar{P}}^G$, where $\bar{P}$ is the parabolic opposite to P.*

Theorem 3.4.

(i) *Any irreducible G-representation is admissible.*
(ii) *(Howe) Any finitely generated, admissible representation is of finite length.*
(iii) *If σ is irreducible then $\operatorname{Ind}_P^G \sigma$ is of finite length. If π is irreducible, then π_N is of finite length.*

Remark 3.5. *(Normalised induction) If Γ is any locally compact topological group, it has a left Haar measure $d_l\gamma$ which is unique up to scalar. For any $x \in \Gamma$, $d_l\gamma(\gamma x^{-1})$ is another left Haar measure, so there is $\delta_\Gamma(x) \in \mathbb{R}_+^\times$ such that $d_l\gamma(\gamma x^{-1}) = \delta_\Gamma(x) d_l\gamma(\gamma)$. The map $\delta_\Gamma : \Gamma \to \mathbb{R}_+^\times$ obtained is a homomorphism and is called the* modulus character*. It is easily seen to be trivial on any compact subgroup, so that it is smooth if Γ contains a compact open subgroup. It is trivial iff G has a bi-invariant Haar measure. This happens, for example, if $\Gamma = G$ reductive or if Γ compact.*

If σ is a smooth representation of M, the normalised parabolic induction is $\operatorname{Ind}_P^G(\sigma\delta_P^{1/2})$. It has the convenient property that it is unitarisable (i.e., there is a G-invariant positive-definite hermitian inner product) if σ is. Note that δ_P has a simple description in terms of roots.

Similarly, the normalised Jacquet module is $J_N(\pi) \otimes \delta_P^{-1/2}$ (this is to make "normalised Frobenius reciprocity" hold).

Example 3.6. *(Principal series for GL_n) Consider $G = GL_n(F) \supset B \supset T$, where B is the Borel subgroup of upper-triangular matrices and T the diagonal torus. Let $\chi = \chi_1 \otimes \cdots \otimes \chi_n : T \to \mathbb{C}^\times$ be a smooth character. Note that the modulus character is $\delta_B(t_1, \ldots, t_n) = |t_1|^{n-1}|t_2|^{n-3} \cdots |t_n|^{-(n-1)}$. Then the* principal series *representation $\operatorname{Ind}_B^G(\chi\delta_B^{1/2})$ has the following properties*

(i) *It has length at most $n!$, the order of the Weyl group.*

(ii) *Its semisimplification is independent of the order of the* χ_i.

(iii) *(Bernstein-Zelevinsky) It is irreducible iff* $\chi_i\chi_j^{-1} \neq |.|$ *for all* i, j.

(In fact the analogues of (i) and (ii) hold for general G*.) The unramified principal series—when each* χ_i *factors through the absolute value—play a special role in the theory of automorphic representations; see later talks.*

Exercise 8. *(Intertwining operators for* GL_2*) Given* $\chi = \chi_1 \otimes \chi_2 : T \to \mathbb{C}^\times$ *with* $|\chi_1(\varpi)| < |\chi_2(\varpi)|$. *Show that if* $f \in \mathrm{Ind}_B^G(\chi\delta_B^{1/2})$,

$$\int_F f\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx$$

is absolutely convergent for any $g \in G$ *and gives a non-zero map from* $\mathrm{Ind}_B^G((\chi_1 \otimes \chi_2)\delta_B^{1/2})$ *to* $\mathrm{Ind}_B^G((\chi_2 \otimes \chi_1)\delta_B^{1/2})$. *(Hint: use the local constancy of* f *near infinity.) This shows where the modulus character comes from. There is a beautiful meromorphic continuation argument to extend to the case of arbitrary* χ_i *(with poles when* $\chi_1 = \chi_2$*); see Bump's book, §4.5.*

Example 3.7. *(Steinberg representation) Let* $B \subset G$ *be a Borel subgroup. Then the* Steinberg representation *is defined as follows:*

$$St_G := \mathrm{Ind}_B^G \mathbb{1} / \sum_{B \subsetneq P} \mathrm{Ind}_P^G \mathbb{1},$$

where the sum runs over all parabolic subgroups containing B *(including* G*). It is known to be irreducible.* St_G *can be viewed as the space of locally constant complex-valued functions on the flag variety* $B\backslash G$ *modulo those functions that factor through one of the quotient partial flag varieties* $P\backslash G$. *In particular if* $G = GL_2(F)$, St_G *is the space of locally constant functions on* $\mathbb{P}^1(F)$ *modulo constant functions.*

See also the longer exercise on p. 10.

4. Various classes of representations

Exercise 9. *If* π *is irreducible smooth (hence admissible), show that the centre of* G *acts by a character. This is called the* central character *of* π.

Definition 4.1. *If* π *is a smooth representation, its* matrix coefficients *are the locally constant functions* $m_{\widetilde{v},v}(g) = \widetilde{v}(gv)$ *(for* $\widetilde{v} \in \widetilde{\pi}$, $v \in \pi$*). This obviously generalises the* (i,j)*-matrix coefficients of a finite-dimensional representation.*

Definition 4.2. *An irreducible smooth representation* π *with unitary central character is* supercuspidal *if its matrix coefficients are compactly supported modulo the centre* $Z = Z(G)$.

(It is non-standard to demand that the central character be unitary, but it will be more convenient in the following.)

By a theorem of Jacquet, this is equivalent to π not being a subrepresentation of $\mathrm{Ind}_P^G \sigma$ for σ an admissible representation of a Levi subgroup M

of a proper parabolic P. (In fact one can replace "subrepresentation" by "subquotient.") By Frobenius reciprocity this is equivalent to $J_N(\pi) = 0$ for all proper parabolics $P = MN$.

Thus supercuspidals are the basic building blocks. They are very hard to construct, the only simple example for $GL_n(F)$ are the characters when $n = 1$. Explicit examples for $GL_2(F)$ will appear in Frank Calegari's talk.

Remark 4.3. *If π is irreducible smooth, then $\pi \hookrightarrow \operatorname{Ind}_P^G \sigma$ for some parabolic subgroup P, possibly equal to G, and some admissible representation σ of M. Since $J_N(\pi)$ has finite length* (3.4)*, we may assume σ to be irreducible, and by transitivity of induction (and mathematical induction) σ can be taken to be supercuspidal. It is known that the pair (M, σ) (with σ supercuspidal and $\pi \hookrightarrow \operatorname{Ind}_P^G \sigma$) is unique up to conjugation. It is called the* supercuspidal support *of π.*

Definition 4.4.

(i) *An irreducible smooth representation π with unitary central character is said to be a* discrete series representation *(or square integrable) if its matrix coefficients lie in $L^2(G/Z)$. (Note that the central character of absolute value 1 allows to integrate over G/Z.)*

(ii) *An irreducible smooth representation π with unitary central character is said to be* tempered *if its matrix coefficients lie in $L^{2+\epsilon}(G/Z)$ for all $\epsilon > 0$.*

Exercise 10. *Show that a discrete series representation is unitarisable. (Hint: use matrix coefficients for a fixed, non-zero $\widetilde{v} \in \widetilde{V}$.) Deduce that its matrix coefficients are bounded.*

Thus supercuspidal $\Rightarrow$ discrete series $\Rightarrow$ tempered (the latter follows from the exercise). The basic example of a discrete series representation that is (generally) not supercuspidal is the Steinberg representation.

In fact there is a completely algebraic characterisation of discrete series and tempered representations in terms of "exponents." If π is irreducible smooth and $P = MN$ any parabolic, $J_N(\pi)$ has finite length so there are smooth characters $\chi_i : Z(M) \to \mathbb{C}^\times$ (not necessarily distinct) such that $\prod_{i=1}^r (\chi_i(z) - z)$ annihilates $J_N(\pi)$. The restriction of the $|\chi_i|$ to the connected component A_M of $Z(M)$ (which is a torus) times $\delta_P^{-1/2}$ are called the (normalised) *exponents* of π. These are important as they control the growth of the matrix coefficients. Note that the real vector space $\operatorname{Hom}_{cts}(A_M, \mathbb{R}_+^\times) \cong X(A_M) \otimes \mathbb{R}$ naturally contains the free abelian subgroup spanned by the roots of A_M (for its action on $\operatorname{Lie} G$).

In another direction, any admissible representation π has a character distribution $\operatorname{tr} \pi : C_c^\infty(G) \to \mathbb{C}$. Here a compactly supported, smooth function $f : G \to \mathbb{C}$ acts on $v \in \pi$ by $fv = \int_G f(g) gv \, dg$, and its trace is well defined by admissibility. There exists a convolution algebra $\mathcal{C}(G)$ of Schwartz–Harish-Chandra functions on G containing $C_c^\infty(G)$, analogous to the classical algebra of Schwartz functions. It carries a natural topology. A distribution $C_c^\infty(G) \to \mathbb{C}$ is said to be *tempered* if it extends to a continuous linear map $\mathcal{C}(G) \to \mathbb{C}$. (See Waldspurger's notes for the definition of $\mathcal{C}(G)$.)

Finally, Harish-Chandra's main result, the Plancherel theorem, which he first established for real reductive groups, is a huge generalisation of the classical Fourier theory, in particular the Fourier inversion theorem/Plancherel formula: it expresses a sufficiently nice (e.g., Schwartz) function f in terms of its "Fourier coefficients." It turns out that precisely the tempered representations contribute to the decomposition. If π is tempered, the Fourier coefficient of f w.r.t. π is given by

$$f_\pi(g) = \operatorname{tr} \pi(f(-g)).$$

(Note that $f(-g)$ is the function sending h to $f(hg)$.) The theorem then says that there is a measure $d\mu$ on the set of irreducible representations $\widehat{G}$ of G (supported on the set of tempered representations) such that for f a Schwartz–Harish-Chandra function,

$$f(g) = \int_{\widehat{G}} f_\pi(g)\, d\mu(\pi).$$

(It is very instructive to think about the case of finite or compact groups, where the Fourier coefficients are given by the same formula and $d\mu(\pi) = \deg \pi$. The key input in these cases is Schur orthogonality.) There also is a version for fixed central character.

The following statements for an irreducible smooth representation π are equivalent:

- π is a discrete series representation.
- For any parabolic subgroup $P = MN$, the exponents of π w.r.t. P lie in the open cone spanned by the positive roots of A_M.
- The subset $\{\pi\} \subset \widehat{G}$ has positive Plancherel measure (for the Plancherel theorem of fixed central character).

Likewise the following are equivalent:

- π is tempered.
- For any parabolic subgroup $P = MN$, the exponents of π w.r.t. P lie in the closed cone spanned by the positive roots of A_M.
- π is a subquotient (equivalently, direct summand) of the normalised parabolic induction of a discrete series representation.
- $\operatorname{tr} \pi$ is a tempered distribution.
- π lies in the support of the Plancherel measure.

By the third characterisation (found in Waldspurger's notes), we see that tempered $\Rightarrow$ unitarisable.

5. Classification

We have seen that

$$\widehat{G}_{\mathrm{sc}} \subset \widehat{G}_{\mathrm{ds}} \subset \widehat{G}_{\mathrm{temp}} \subset \widehat{G}_{\mathrm{unit}} \subset \widehat{G},$$

where $\widehat{G}$, the dual of G, is the set of irreducible smooth G-representations as before and $\widehat{G}_{\mathrm{sc}}$ (resp., $\widehat{G}_{\mathrm{ds}}$, $\widehat{G}_{\mathrm{temp}}$, $\widehat{G}_{\mathrm{unit}}$) is the subset of supercuspidal (resp., discrete series, tempered, unitarisable) representations.

Let us give some indications of what is known about the classification of $\widehat{G}$. Note that this is still considerably weaker than establishing a local Langlands correspondence. For $GL_n(F)$ the classification is known (in particular all the steps mentioned below) due to work of Bernstein-Zelevinsky and Bushnell-Kutzko, whereas the natural matching of supercuspidals with Galois data required completely new (global) methods.

The philosophy, which as far as I know comes from the work of Harish-Chandra and Langlands (and first in the case of real groups), is that tempered representations should be classified in terms of discrete series and general irreducible representations in terms of tempereds. In fact the first comes down by the equivalent characterisations of temperedness to decomposing normalised inductions of discrete series (and there seems to be a partial theory of the "R-group") and the second is known and called the Langlands classification (due to Silberger in the p-adic case).

Interestingly, the classification of unitary representations is extremely hard (also in the real case) and mostly known for $GL_n(F)$ (Tadić) and $GL_n(D)$ where D/F is a division algebra (finished by Sécherre) at this point.

For the construction of discrete series in terms of supercuspidals, this is completely known for $GL_n(F)$ and seems to be mostly known for classical groups by work of Mœglin and Tadić (relying on a reducibility hypothesis of certain representations which is implied by Arthur's conjectures).

Finally, there is a folklore conjecture regarding supercuspidal representations.

Conjecture 5.1. *Any supercuspidal representation π is of the form* $\text{c-Ind}_U^G \tau$*, where U is an open compact-mod-centre subgroup and τ is an irreducible (hence finite-dimensional) representation of U.*

This is known to be true for $GL_n(F)$ by an explicit construction of all supercuspidals by Bushnell-Kutzko (1993). The main difficulty occurs for some exceptional values of p ($p \mid n$ in the case of $GL_n(F)$). For general $\mathbf{G}$ there is a construction of supercuspidals due to Jiu-Kang Yu (2001); this was shown to be exhaustive for p sufficiently large by Ju-Lee Kim (2007).

6. Further exercise

Exercise 11. *(Principal series for $GL_2(\mathbb{Q}_p)$, after Matt Emerton)*

(i) *(Steinberg) The map $f \mapsto f(\infty)$ gives a B-equivariant short exact sequence*

$$0 \to C_c^\infty(\mathbb{Q}_p) \to C^\infty(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathbb{C} \to 0.$$

Note that B acts on $C_c^\infty(\mathbb{Q}_p)$ by translations and by scalings and show that $C_c^\infty(\mathbb{Q}_p) \cong St$ (as smooth B-representations). Suppose $0 \neq \phi \in C_c^\infty(\mathbb{Q}_p)$ and let M be the B-submodule generated by ϕ.

Show that there is an element $\phi' \in M$ that is invariant under translations by $\mathbb{Z}_p$ and such that $\phi'(0) \neq 0$. By averaging over $\mathbb{Z}_p^\times$, show that there is an element $\phi'' \in M$ that is moreover invariant under scalings by $\mathbb{Z}_p^\times$.

Note that the Hecke operator $h : \phi \mapsto (x \mapsto \sum_{a \in \mathbb{Z}_p/p\mathbb{Z}_p} \phi(\frac{x+a}{p}))$ preserves M and sends ϕ'' to an element with the same properties and of strictly smaller support, unless ϕ'' is a scalar multiple of either $[\mathbb{Z}_p]$ or $[\frac{1}{p}\mathbb{Z}_p] - p[\mathbb{Z}_p]$. Deduce that $C_c^\infty(\mathbb{Q}_p)$ has precisely one non-trivial B-submodule: the kernel of $\phi \mapsto \int_{\mathbb{Q}_p} \phi$.

Finally show that St is an irreducible G-module.

(ii) *Show that a slight adaptation of the same argument goes through over any field of characteristic p, with the difference that St is irreducible even as a B-module.*

(iii) *Use the same argument to show that $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ is irreducible, unless $\chi_1\chi_2^{-1}$ equals 1 or $|.|^2$ and there is exactly one non-trivial G-submodule. It's useful to note that the kernel of $f \mapsto f(1)$ (as in (i)) is still isomorphic to $C_c^\infty(\mathbb{Q}_p)$ by letting $\phi(x) = f\left(\begin{pmatrix} & 1 \\ 1 & x \end{pmatrix}\right)$, but the B-action depends on the χ_i. It's still true that $C_c^\infty(\mathbb{Q}_p)$ has a unique non-trivial B-submodule.*

7. Useful References

- Cartier notes "Representations of p-adic groups: a survey" (Corvallis); gives a good overview with sketches of some proofs
- Bump book "Automorphic Forms and Representations" (CUP); proves interesting results in the special case of $GL_2(F)$
- Casselman's unpublished notes "Introduction of the theory of admissible representations of p-adic reductive groups", available at `http://www.math.ubc.ca/people/faculty/cass/research.html`; very well written with detailed proofs
- Bernstein-Zelevinsky "Representations of the group $GL(n, F)$, where F is a local non-Archimedean field" (1976); concise and well written; focuses on $GL_n(F)$; unusual notations
- Waldspurger "La formule de Plancherel pour les groupes p-adiques d'après Harish-Chandra" (2003); a very careful write-up of Harish-Chandra's proof of the Plancherel formula in modern notation

8. Acknowledgements

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