Here are some practice problems in recurrence relations. The first 9 problems (roughly) are basic, the other ones are competition-level.

1. Find a formula for F_n, where F_n is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$.

2. Fix any positive integer k. Show that there are infinitely many Fibonacci numbers divisible by k.

3. Find a formula for a_n, where $a_0 = a_1 = 2$, $a_{n+1} = 4a_n - 4a_{n-1}$.

4. Find a 3-term recurrence relation for the sequence $a_n = 3^{n+1} - 2 \cdot 5^n$. Now do the same for $a_n = 3^{n+1} - 2 \cdot 5^n + n^2$.

5. Find $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$.

6. Suppose that $a_0 = 0$, $b_0 = 1$ and that $a_n = a_{n-1} + 2b_{n-1}$, $b_n = -a_{n-1} + 4b_{n-1}$. Find formulas for a_n and b_n.

7. Suppose that $x_0 = 18$, $x_{n+1} = \frac{10}{3}x_n - x_{n-1}$, and that the sequence $(x_n)_{n=1}^{\infty}$ converges (to some real number). What is x_1?

8. Show for positive integers m, n: $m \mid n$ if and only if $F_m \mid F_n$ (where $(F_n)_n$ is the Fibonacci sequence).

9. Suppose Bob keeps throwing a coin and each time scores one point for a head and two points for a tail. For a fixed $n \geq 1$, what is the probability that his score is precisely n points at some point? (Hint: let p_n be the probability this happens, and try to find a recurrence relation. Maybe wait until the probability session.)

10. Suppose $a_0 = 0$, $a_1 = 2$ and $a_{n+2} = 4a_{n+1} - 4a_n + n^2 - 5n + 2$. Show that n divides a_n for all $n \geq 1$.

11. Let $a_n = \lfloor (5 + \sqrt{21})^n \rfloor + 1$, so $a_0 = 2$, $a_1 = 10$, $a_2 = 92$, ... Prove that a_n is divisible by 2^n.

12. Find the digit immediately to the left and right of the decimal point for $(\sqrt{7} + \sqrt{13})^{2008}$.

13. Write $(2 + \sqrt{3})^{2n-1} = a_n + b_n\sqrt{3}$ for integers a_n and b_n ($n \geq 1$). Show that $a_n - 1$ is a perfect square.

14. Let
$$T_0 = 2, T_1 = 3, T_2 = 6,$$
and for $n \geq 3$,
$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences. (A-1, Putnam 1990)

15. Define a **selfish** set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \dots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish. (B-1, Putnam 1996)

16. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \dots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$. (A-2, Putnam 1993)

17. Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that
$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$
for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.) (A-3, Putnam 2004)

18. Let $1, 2, 3, \dots, 2005, 2006, 2007, 2009, 2012, 2016, \dots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \dots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006. (A-3, Putnam 2006)

19. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5}\rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.) (B-3, Putnam 2007)

20. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set
$$\left\{\frac{a_{n-1}}{a_n} : n \geq 1\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots\right\}.$$
(A-5, Putnam 2002) (Note: you will need some number theory we didn't discuss two weeks ago.)

21. The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \geq 4$,
$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$
Show that, for all n, a_n is an integer multiple of n. (A-6, Putnam 1999)