Here are some practice problems in number theory. The earlier problems tend to be easier.

1. Show that the sequence 11, 111, 1111, ... does not contain any perfect square.

2. Show that $2^{70} + 3^{70}$ is divisible by 13.

3. How many zeros are at the end of $2008! = 1 \cdot 2 \cdots 2007 \cdot 2008$?

4. (a) Find all natural numbers n such that $7 \mid 2^n - 1$.

   (b) Show that there is no natural number n such that $7 \mid 2^n + 1$.

5. Find the last two digits of $7^{7^{\cdot^{\cdot^{7}}}}$, where the tower consists of seven 7's.

6. If $\gcd(a, b) = 1$ show that $\gcd(a-b, a+b) \leq 2$ and $\gcd(a^2-ab+b^2, a+b) \leq 3$.

7. Let $a_n = 10 + n^2$ for $n \geq 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers. (If you can do that, what about $a_n = d + n^2$, where d is any natural number?)

8. Show there exist three consecutive integers, each of which is divisibly by the 100th power of an integer bigger than 1.

9. (a) Show that there are infinitely many primes of the form $6n-1$. (Hint: if there are only finitely many, $p_1, \ldots, p_r$, consider $6(p_1 \ldots p_r)^2 - 1$ and obtain a contradiction.)

   (b) Show that there are infinitely many primes of the form $4n - 1$.

10. A triangular number is a positive integer of the form $n(n+1)/2$. Show that m is a sum of two triangular numbers iff $4m + 1$ is a sum of two squares. (A-1, Putnam 1975)

11. Suppose $n > 1$ is an integer. Show that $n^4 + 4^n$ is not prime.

12. How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1? (A-1, Putnam 1989)

13. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.) (A-1, Putnam 2005)

14. For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i? (B-1, Putnam 1991)

15. d, e and f each have nine digits when written in base 10. Each of the nine numbers formed from d by replacing one of its digits by the corresponding digit of e is divisible by 7. Similarly, each of the nine numbers formed from e by replacing one of its digits by the corresponding digit of f is divisible by 7. Show that each of the nine differences between corresponding digits of d and f is divisible by 7. (A-3, Putnam 1993)

16. Define the sequence of decimal integers a_n as follows: $a_1 = 0$; $a_2 = 1$; a_{n+2} is obtained by writing the digits of a_{n+1} immediately followed by those of a_n. When is a_n a multiple of 11? (A-4, Putnam 1998)

17. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \dots \binom{p}{k}$$

of binomial coefficients is divisible by p^2. (A-5, Putnam 1996)

18. Find all positive integers a, b, m, n with m relatively prime to n such that $(a^2 + b^2)^m = (ab)^n$. (A-3, Putnam 1992)

19. Suppose the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

20. Find all solutions of $x^{n+1} - (x + 1)^n = 2001$ in positive integers x and n. (A-5, Putnam 2001)