Putnam Questions – Week 7

One general strategy for such problems: try to experiment with small values of n and look for patterns.

1. Alice and Bob play the following game. They start with a pile of 9 matches. They take turns, Alice playing first. Each player may remove between 1 and 3 matches. The player who picks up the last match wins. Who has a winning strategy? And what is it?

2. Same problem as before, but the player who takes the last match loses. Who has a winning strategy now? And what is it?

3. In both of the above problems, what if the players start with n matches? That is, who has a winning strategy, depending on n?

4. Another variation of the first problem: now the players may take 2^d matches for any $d \geq 0$. If they start with n matches and the player who takes the last match wins, who has a winning strategy and what is it? (depending on n)

5. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

6. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

7. Suppose there are n lights in a row. Each is equipped with a switch, which changes that lamp *and its neighbouring lamps* from on to off or vice versa. (So operating the switch at one of the ends changes two lamps, any other switch changes three lamps.) For which n is it guaranteed that all lamps can be switched off, no matter which lamps are on initially.

8. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)