Putnam Questions – Week 6

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x)f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

2. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

3. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

4. Find all real-valued continuously differentiable functions f on the real line such that for all x,
$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

5. Evaluate
$$\int_2^4 \frac{\sqrt{\ln(9-x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$
[Hint: try to exploit symmetry (antiderivative is hopeless).]

6. Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,
$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$
If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.
[Hint: try to find an expression for f' and g'...]

7. For each continuous function $f : [0,1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\, dx$ and $J(x) = \int_0^1 x\,(f(x))^2\, dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.