Putnam Problems: Algebra (F.H., Fall 2016)

November 10, 2016

A1-2001 Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

A1-1995 Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

A2-2012 Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x*z = y$. (This z may depend on x and y.) Show that if a, b, c are in S and $a*c = b*c$, then $a = b$.

A2-2014 Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \leq i, j \leq n$. Compute $\det(A)$.

A2-1991 Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

B2-1968 A is a subset of a finite group G, and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

B2-1989 Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

B3-1979 Let F be a finite field with n elements, where n is odd, and suppose that $p(x) := x^2 + bx + c$ ($b, c \in F$) is an irreducible polynomial over F. For how many elements $k \in F$ is $p(x) + k$ irreducible? (You don't need to know about the theory of finite fields for this question, it's enough to know what a field is.)

B3-1972 Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove $B = 1$.

A4-1997 Let G be a group with identity e and $\phi : G \to G$ a function such that

$$\phi(g_1)\phi(g_2)\phi(g_3) = \phi(h_1)\phi(h_2)\phi(h_3)$$

whenever $g_1g_2g_3 = e = h_1h_2h_3$. Prove that there exists an element $a \in G$ such that $\psi(x) = a\phi(x)$ is a homomorphism (i.e. $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$).

Please let me know if there are any typos! You can find a lot more algebra problems at

`www.math.utoronto.ca/barbeau/putnamgp.pdf`,
`www.math.utoronto.ca/barbeau/putnamalg.pdf`,
`www.math.utoronto.ca/barbeau/putnamla.pdf`.