Classification of irred. mod p representations of a p-adic GL_n.

Galois Trimester (IHP)
1/4–6/10

Motivation:

$F/\mathbb{Q}_p$ fin.

$\mathcal{O} = \mathcal{O}_F$

$k = \mathcal{O}/(\varpi)$, $|k| = q$.

$G = GL_n$ or $GL_n(F)$

Hope: Mod p local Langlands

$$\left\{ \begin{array}{l} \text{cts. homos.} \\ \rho : \mathrm{Gal}(\bar{F}/F) \longrightarrow \hat{G}(\bar{k}) \end{array} \right\} /\cong \quad \overset{??}{\longleftrightarrow} \quad \left\{ \begin{array}{c} \underline{\text{some}} \text{ adm. smooth} \\ G\text{-reps. over } \underset{\overline{\mathbb{F}}_p}{\overset{\shortparallel}{\bar{k}}} \end{array} \right\} /\cong$$

+ compatible with p-adic local Langlands.

$\underline{GL_2(\mathbb{Q}_p)}$: Breuil, Berger, Colmez

ρ red $\longleftrightarrow$ extension of two PS.

ρ irr. $\longleftrightarrow$ supersing.

Known about irr. adm. reps.:

$n = 2$: Barthel-Livné, $\underbrace{\text{Breuil, Paskunas, B–P, Hu}}_{\text{ss.}}$

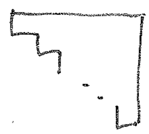
$n > 2$: $\underbrace{\text{Ollivier}}_{\text{PS}}$, $\underbrace{\text{Vignéras}}_{\text{Steinberg}}$, $\underbrace{\text{Große-Klönne}}_{\text{gen}^{d}\text{ Steinberg}}$.

Main result:

Thm. 1 (H.) The irred. adm. G-reps. over $\bar{k}$ are of the form

$\mathrm{Ind}_{\bar{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$, where:

(i) P std. parab.



$n_1\ n_2 \cdots n_r$

(ii) σ_i irr. adm. $GL_{n_i}(F)$-rep.

either: σ_i supersing., $n_i \geq 2$

or: $\sigma_i = \sigma_i' \otimes (\eta_i \circ \det)$

σ_i': constit. of $\mathrm{Ind}_{\bar{B}}^{G}(1)$ (gen^d Steinberg)

$\eta_i : F^\times \to \bar{k}^\times$ smooth

(iii) $\eta_i \neq \eta_{i+1}$ $\forall i$.

Moreover, $(P, (\sigma_i))$ is uniquely def^d.

Cor.: (a) Can decompose $\mathrm{Ind}_{\bar{P}}^{G}\sigma$ into irreds. (f.l. + mult. 1)

σ: irr. adm.

(b) π irred.: adm. supersing = supercusp.

supercusp.: (not constit in (a), $P \neq G$)

(c) π irred. adm. has "constant Hecke evals."

<u>Background</u>: All reps. will be over $\bar{k} \cong \bar{\mathbb{F}}_p$!

$$G = GL_n(F)$$
$$\cup$$
$$K = GL_n(\mathcal{O}) \quad \text{max. cpt.}$$
$$\cup$$
$$K(1) = 1 + \varpi M_n(\mathcal{O}) \quad \text{pro-}p.$$

π ... G-rep.

<u>Recall</u>: π <u>smooth</u> iff $\pi = \bigcup_{U \text{ open subgp.}} \pi^U$

π <u>adm</u> if $\dim \pi^U < \infty \quad \forall U$ open subgp.

$(\Leftrightarrow \dim \pi^{K(1)} < \infty)$.
from Lemma.

<u>Lemma 1</u>: τ ... non-0 smooth rep. of a pro-p gp. H over char. p field E.

$\Rightarrow \tau^H \neq 0$.

<u>Pf</u>: τ smooth $\Rightarrow$ wlog. H is finite p-gp.

$\Rightarrow$ any $x \in \tau$ is contained in a <u>finite</u> $\mathbb{F}_p H$-submod.

$\Rightarrow$ wlog. $E = \mathbb{F}_p$ and $|\tau| < \infty$.

$$|\tau| = \sum \underbrace{|\text{Orb}|}_{p\text{-power}} \Rightarrow |\tau^H| \equiv 0 \ (p). \qquad \square$$

<u>Cor. 1</u>: Any irr. smooth K-rep. V factors through $K/K(1) = G(k)$.

<u>Pf</u>: consider $V^{K(1)} \neq 0$. $\square$

Def.: A weight is an irr. $G(k)$-rep. (up to iso.) fin. many!

Cor. 2: π ... non-zero sm. G-rep. $\Rightarrow$ $\pi|_K$ contains a weight.

Pf: $\pi^{K(1)} \neq 0$ contains an irr. subrep. $\square$

$\circlearrowright$

$G(k)$

Ex.: $n=2$, $k=\mathbb{F}_p$. Weights: $\underbrace{\mathrm{Sym}^{a-b}\,\bar{k}^2 \otimes \det{}^b}_{F(a,b)}$, $0 \le a-b \le p-1$.

In general:

$F(a_1, \dots, a_n)$ ... irr. GL_n-rep. of h.w. $(a_1, \dots, a_n)$ restricted to $G(k)$. (Steinberg)

$0 \le a_i - a_{i+1} \le q-1 \ \forall i$

$F(\underline{a} + \underline{1}) \cong F(\underline{a}) \otimes \det$.

So there are $q^{n-1}(q-1)$ wts.

Hecke action

π ... adm. G-rep.

V ... any wt. of π: $V \subset \pi|_K$.

cpt. ind.: $\mathrm{ind}_K^G V := \{ \varphi : G \to V : \varphi(kg) = k\varphi(g) \ \forall k \in K, g \in G$, $\mathrm{supp}\, \varphi$ cpt. $\}$.

$\cong \bar{k}G \otimes_{\bar{k}K} V$.

$$\rightsquigarrow \quad \mathrm{Hom}_K(V, \pi) = \mathrm{Hom}_G(\mathrm{ind}_K^G V, \pi)$$

$$\circlearrowleft$$

$$\mathcal{H}_G(V) := \mathrm{End}_G(\mathrm{ind}_K^G V)$$

<u>Lemma 2</u>: $\mathcal{H}_G(V) \cong \{ \varphi : G \to \mathrm{End}_{\bar{k}} V : \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2, \ \mathrm{supp}\, \varphi \text{ cpt.} \}$.

$$(\varphi_1 * \varphi_2)(g) = \sum_{G/K} \varphi_1(g\gamma)\varphi_2(\gamma^{-1})$$

In part., if $V = \mathbb{1}$: $\underbrace{\mathcal{H}_G(\mathbb{1}) = \bar{k}[K \backslash G / K]}_{\text{unram. Hecke alg.}} \curvearrowright \mathrm{Hom}_K(\mathbb{1}, \pi) = \pi^K$

$\underbrace{\mathcal{H}_G(V)}_{\substack{\text{comm.} \\ (\to \text{later})}} \curvearrowright \underbrace{\mathrm{Hom}_K(V, \pi)}_{\substack{\text{f.d.} \\ \text{as } \pi \text{ adm.}}}$, so it's a direct sum of gen'd eigenspaces.

If $f : V \hookrightarrow \pi$ is an evec:

$$T(f) = \chi(T) f \qquad \forall T \in \mathcal{H}_G,$$
$$\chi : \mathcal{H}_G \to \bar{k} \text{ alg. homo.}$$

$\rightsquigarrow \tilde{f} : \mathrm{ind}_K^G V \longrightarrow \pi$ G-lin.

$\rightsquigarrow \tilde{\tilde{f}} : \mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi \longrightarrow \pi$

<u>Upshot</u>: $\boxed{\begin{array}{l} \text{Hecke evals. } \chi \text{ occur on mult. space } \mathrm{Hom}_K(V, \pi) \\ \Leftrightarrow \ \exists \ \mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi \xrightarrow[\neq 0]{} \pi. \end{array}}$

(<u>quotient</u> of π irr.)

$G = GL_1$:

irr. adm. repr.: $F^\times \longrightarrow \bar{k}^\times$ smooth

triv. on $K(1) = 1 + \pi\mathcal{O}$

$$\mathrm{Hom}_{sm}(F^\times, \bar{k}^\times) \cong \mathbb{Z}/(q-1) \times \bar{k}^\times .$$

$$\psi \longmapsto \underbrace{\psi|_{\mathcal{O}^\times}}_{\text{weight}},\ \psi(\varpi).$$

$G = GL_2$ (Barthel–Livné):

$\mathcal{H}_G(V) \cong \bar{k}[T_1, T_2^{\pm 1}]$ (→ later)

supp $T_1 = K\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}K$, supp $T_2 = K\begin{pmatrix}\varpi & \\ & \varpi\end{pmatrix}K$.

$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi$: $V = F(a,b)$, $0 \le a-b \le q-1$.

(1) $\chi(T_1) \neq 0$:

up to semi-simplification, get princ. series $\mathrm{Ind}_{\bar{B}}^G(\chi_1 \otimes \chi_2)$, $\bar{B} = \begin{pmatrix} * & \\ * & * \end{pmatrix}$

where:

$\chi_1(x) = \bar{x}^a$, $\chi_2(x) = \bar{x}^b$ $(x \in \mathcal{O}^\times)$.

$\chi_2(\varpi) = \chi(T_1)^{-1}$ (not hard: ∃ map ...)

$\chi_1(\varpi)\chi_2(\varpi) = \chi(T_2)^{-1}$.

$\mathrm{Ind}_{\bar{B}}^G(\chi_1 \otimes \chi_2)$ ↗ irr. if $\chi_1 \neq \chi_2$

$\mathrm{Ind}_{\bar{B}}^G(\chi_1 \otimes \chi_2)$ ↘ length 2 if $\chi_1 = \chi_2$. Wlog. $\chi_1 = \chi_2 = 1$.

non-split

$$0 \to 1 \to \mathrm{Ind}_{\bar{B}}^G(1) \to \underset{\substack{\text{irred.} \\ \text{(Steinberg)}}}{St} \to 0$$

ind $V \otimes \chi$: $\overset{\text{extension}}{\text{other way}}$ $\nexists$ $\dim V = 1$.

② $\underline{\chi(T_1) = 0}$:

$F = \mathbb{Q}_p$: $\underline{\text{irr.}}$ (Breuil) each has 2 wts.

$F \neq \mathbb{Q}_p$: in many cases (always?) it has ∞ many irr. quot. (B – P).

} "supersing".

<u>Conclusion:</u>

irr. adm. reps of $GL_2(F)$:

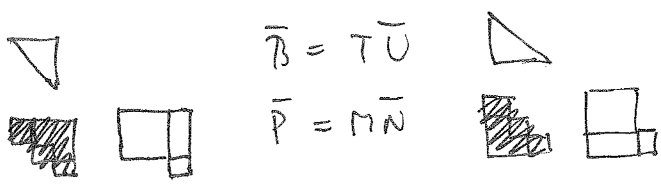
- 1-dim $\eta \circ \det$
- $St \otimes (\eta \circ \det)$
- PS $\mathrm{Ind}_{\bar{B}}^{G}(\chi_1 \otimes \chi_2)$, $\chi_1 \neq \chi_2$
- supersing.

(disjoint)

<u>Satake iso.</u>

$B = TU$ Borel. $\bar{B} = T\bar{U}$

$P = MN$ std. parab. $\bar{P} = M\bar{N}$



V ... weight

<u>Lemma 3</u> (Smith, Cabanes)

$V^{N(k)}$ & $V_{\bar{N}(k)}$ are irred. as $M(k)$-rep ($\Rightarrow$ weights w.r.t. M.)

The nat. $M(k)$-lin. map $V^{N(k)} \hookrightarrow V \twoheadrightarrow V_{\bar{N}(k)}$ is an iso.

<u>Ex.</u> : $F(a,b,c)^{\begin{pmatrix}1 & * & * \\ & 1 & \\ & & 1\end{pmatrix}} \cong F(a) \otimes F(b,c)$.

$$\mathcal{H}_G(V) = \{ \varphi : G \longrightarrow \operatorname{End} V : \ \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2, \ \operatorname{supp}\varphi \text{ cpt.} \}.$$

<u>As vector space:</u>

$$G = \coprod_{\lambda \in Y(T)_-} K\lambda(\varpi)K = \coprod_{i_1 \le \dots \le i_n} K \begin{pmatrix} \varpi^{i_1} & & \\ & \ddots & \\ & & \varpi^{i_n} \end{pmatrix} K \qquad \text{(Cartan)}$$

Fix $\lambda \in Y(T)_-$

$\varphi \in \mathcal{H}_G(V)$, $\operatorname{supp}\varphi \subseteq K\underbrace{\lambda(\varpi)}_{=:t}K$

$$k \circ \varphi(t) = \varphi(kt) = \varphi(t) \circ (t^{-1}kt) \qquad \forall k \in K \cap tKt^{-1}.$$

$$\Rightarrow \quad \begin{array}{ccc} V & \xrightarrow{\ \varphi(t)\ } & V \\ \downarrow & & \uparrow \\ V_{N_\lambda(k)} & \dashrightarrow & V^{N_{-\lambda}(k)} \end{array} \quad \Big\} \ M_\lambda(k)\text{-linear.}$$

Lemma 3 $\Rightarrow$ space of such maps is 1-dim^l.

<u>Ex.</u>: $\lambda = (0,0,1)$: $\quad V_{\begin{pmatrix} 1 & & \\ & 1 & \\ * & * & 1 \end{pmatrix}} \dashrightarrow V^{\begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}}$.

<u>Alg-structure:</u>

Satake transform

$$S_G : \mathcal{H}_G(V) \longrightarrow \mathcal{H}_T(\overbrace{V_{\bar{U}(k)}}^{\text{wt. w.r.t. } T \ \Rightarrow \ 1\text{-dim}^l}).$$

$$\varphi \longmapsto \underbrace{\Big(t \longmapsto \sum_{\bar{U}(0)\backslash \bar{U}} \varphi(\bar{u}t)\Big)}_{\in \operatorname{End}(V), \text{ induces map } \in \operatorname{End}(V_{\bar{U}(k)})}$$

$$T^- := \{ t \in T : \ |\alpha(t)| \geq 1 \quad \forall \alpha \text{ pos. root} \} = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} : |t_1| \geq \cdots \geq |t_n| \right\}$$

<u>Thm. 2</u>: S_G is an inj. alg. homo. with image
$$\{ \varphi \in \mathcal{H}_T : \operatorname{supp} \varphi \subset T^- \}.$$

<u>Cor.</u>: $\mathcal{H}_G \underset{\text{choice of unif.}}{\cong} \bar{k}[T^-/T(\mathcal{O})] \cong \bar{k}[Y(T)_-]$. is comm. + noeth.

so $\mathcal{H}_G \cong \bar{k}[\tau_1, \dots, \tau_{n-1}, \tau_n^{\pm 1}]$, where $\tau_i = (0, 0, \dots, \underbrace{1, \dots, 1}_{i}) \in Y(T)_-$.

More generally:

$$S_G^M : \mathcal{H}_G(V) \longrightarrow \mathcal{H}_M(V_{\bar{N}(k)}) \qquad \text{inj. alg. homo.}$$

Transitive:

$$\begin{tikzcd} \mathcal{H}_G \arrow[r, "S_G^M"] \arrow[dr, "S_G"'] & \mathcal{H}_M \arrow[d, "S_M"] \\ & \mathcal{H}_T \end{tikzcd}$$

<u>Parameterising Hecke evals.</u>

Classically,

$$\operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[K \backslash G / K], \mathbb{C}) \overset{1:1}{\longleftrightarrow} \left(\text{unram. chars. } T \longrightarrow \mathbb{C}^\times \text{ triv. on } T(\mathcal{O}) \right) / W$$

Lemma 4:

$\chi \in \mathrm{Hom}_{\bar{k}\text{-alg}}(\mathcal{H}_G(V), \bar{k}) \xleftrightarrow{1:1}$ pairs (M, χ_M) s.t.

- $P = MN$ std. parab.
- $\chi_M : Z_M \longrightarrow \bar{k}^\times$ s.t.

$\chi_M |_{Z_M(\mathcal{O})} =$ centr. char. of $V_{\bar{N}(k)}$.

Also, M is the smallest Levi s.t. χ factors through $\mathcal{H}_M$.

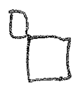
Rk: If $V = \mathbb{1}$, χ_M is an unram. char. of Z_M.

Idea: $(n=3)$

$$\chi : \mathcal{H}_G(V) \longrightarrow \bar{k}$$

$$\mathcal{H}_G(V) \cong \bar{k}[Y(T)_-] = \bar{k}[\tau_1, \tau_2, \tau_3^{\pm 1}].$$

E.g.: $\chi(\tau_1) = 0$, $\chi(\tau_2) \neq 0$:

$\Rightarrow M =$ 

$$\chi_M\begin{pmatrix} 1 & & \\ & \varpi & \\ & & \varpi \end{pmatrix} = \chi(\tau_2)^{-1}$$

$$\chi_M\begin{pmatrix} \varpi & & \\ & \varpi & \\ & & \varpi \end{pmatrix} = \chi(\tau_3)^{-1}.$$

Rk: Will see: π irr. adm.

$\Rightarrow$ all Hecke evals. of π are param^d by ~~same~~ same pair (M, χ_M).

"const. Hecke evals.".

Def. π irr. adm. is <u>supersingular</u> if all Hecke evals. of π are param^d by a pair (M, χ_M) with $\boxed{M = G}$ $\left(\underset{\text{easy}}{\Longrightarrow} \chi_M = \text{central char. of } \pi\right)$

Rk: Equivalently (see above ex.), $\chi(\tau_1) = \dots = \chi(\tau_{n-1}) = 0$.
$\forall$ Hecke evals. χ of π.

So generalises def^n of B-L.

Ex: $n=1$: all irr. adm. reps. are ss.

Lemma 5 (Weights + Hecke evals. in induced reps.)

σ ... ~~smooth~~ adm. M-rep.

V -- weight (w.r.t. G)

$\chi : \mathcal{H}_M(V_{\bar{N}(k)}) \longrightarrow \bar{k}$ alg. hom.

Then

V occurs in $\mathrm{Ind}_{\bar{P}}^{G}\sigma$ with evals. χ $\quad (\chi \circ S_G^M)$

$\Leftrightarrow$ $V_{\bar{N}(k)}$ ------ σ ------ χ.

Cor.: The Hecke evals. in σ and $\mathrm{Ind}_{\bar{P}}^{G}\sigma$ are param^d by the same pairs (L, χ_L). $\quad$ ($\Rightarrow L \subset M$ in each case).

Pf: Frob-rec.

$$\mathrm{Hom}_K(V, \mathrm{Ind}_{\bar{P}}^{G}\sigma) = \mathrm{Hom}_K(V, \mathrm{Ind}_{\bar{P}(\mathcal{O})}^{K}\sigma) \quad \text{as } G = \bar{P}K \text{ (Iwasawa)}$$

$$= \mathrm{Hom}_{\bar{P}(\mathcal{O})}(V, \sigma)$$

$$= \mathrm{Hom}_{M(\mathcal{O})}(V_{\bar{N}(k)}, \sigma).$$

$$\begin{array}{ccc} \circlearrowleft & & \circlearrowleft \\ \mathcal{H}_G(V) & \xrightarrow[S_G^M]{} & \mathcal{H}_M(V_{\bar{N}(k)}) \end{array} \quad \text{(calculation)}$$

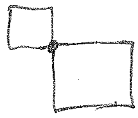
$\square$

(classically: Hecke evals. of unram. PS).

Comparison of cpt. and parab. ind.

Def.: V is <u>M-regular</u> if $\mathrm{Stab}_W(\underbrace{V^{V(k)}}_{\text{1-dim. subspace of } V}) \subset W_M$.

Weyl gp.

Ex.: $M =$  2, 3

$V = F(a, \underline{b, c}, d, e)$ $\qquad$ M-reg. $\Leftrightarrow$ inequ. $b \geq c$ is <u>strict</u>

Ex.: $G = GL_2$, V is T-reg. $\Leftrightarrow$ $\dim V \neq 1$.

Prop. 1: V ... M-reg.

$\chi : \mathcal{H}_M(V_{\bar{N}(k)}) \longrightarrow \bar{k}$ alg. hom.

$$\Rightarrow \mathrm{ind}_K^G V \otimes_{\underbrace{\mathcal{H}_G(V)}_{\text{via Satake}}} \chi \xrightarrow{\sim} \mathrm{Ind}_{\bar{P}}^G (\mathrm{ind}_{M(O)}^M V_{\bar{N}(k)} \otimes_{\mathcal{H}_M(V_{\bar{N}(k)})} \chi).$$

Rk: (i) Such a map exists by Lemma 5.

(ii) $n = 2$, $P = B$: Barthel–Livné

(RHS: princ. series
T-reg. $\Leftrightarrow$ $\dim V \neq 1$, so need this cond^n).

Irreducibility of $\mathrm{Ind}_{\bar{P}}^G (\underbrace{\sigma_1 \otimes \dots \otimes \sigma_r}_{=: \sigma})$, part I

Adm.: $(\mathrm{Ind}\, \sigma)^{K(1)} = \mathrm{Ind}_{\bar{P}(k)}^{G(k)} (\sigma^{M(1)})$ $\qquad$ f.-d. ✓

Irr.: $\quad 0 \neq \pi \subset \mathrm{Ind}_{\bar{P}}^{G} \sigma$ subrep.

π contains some wt. V

Choose $\mathcal{H}_G$-evec. $\quad V \underset{f}{\hookrightarrow} \pi \subset \mathrm{Ind}\,\sigma$, evals.
$\chi: \mathcal{H}_G \to \bar{k}$.

By Lemma 5, χ fact. through $\mathcal{H}_M$ and f corresponds to

$$V_{\bar{N}(k)} \hookrightarrow \sigma, \quad \text{evals. } \chi$$

$$\to \quad \mathrm{ind}_{M(\mathcal{O})}^{M} V_{\bar{N}(k)} \otimes_{\mathcal{H}_M} \chi \twoheadrightarrow \sigma$$

<u>If V is M-reg.:</u>

$$\mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi \underset{\substack{\uparrow \\ \text{Prop. 1}}}{\xrightarrow{\sim}} \mathrm{Ind}_{\bar{P}}^{G} (\text{— — — —}) \underset{\substack{\uparrow \\ \text{as } \mathrm{Ind}_{\bar{P}}^{G} \text{ exact!}}}{\twoheadrightarrow} \mathrm{Ind}_{\bar{P}}^{G} \sigma .$$

LHS is generated by V as G-rep. $\Rightarrow$ so is RHS
$\Rightarrow \pi = \mathrm{Ind}_{\bar{P}}^{G} \sigma$.

<u>In general</u>: need to show that π contains an M-reg. wt. ($\to$ later) //

Maps between cpt. inductions.

V_1, V_2 weights.

<u>Idea</u>: Suppose we know $\mathrm{ind}_K^G V_1 \otimes \chi_1 \cong \mathrm{ind}_K^G V_2 \otimes \chi_2$. $\quad (*)$

Then whenever V_1 occurs in a smooth G-rep. π with Hecke evals χ_1

$\Rightarrow V_2$ — — — — π — — — — χ_2.

So the iso. (*) allows us to "change the weight" from V_1 to V_2 (provided V_1 occurs with evals χ_1).

First: study maps $\mathrm{ind}_K^G V_1 \longrightarrow \mathrm{ind}_K^G V_2$.

$$\mathcal{H}_G(V_1, V_2) := \mathrm{Hom}_G(\mathrm{ind}_K^G V_1, \mathrm{ind}_K^G V_2),$$

a $(\mathcal{H}_G(V_1), \mathcal{H}_G(V_2))$-bimod.

$$\underset{\substack{\uparrow \\ \text{like Lemma 2}}}{\cong} \{\varphi : G \longrightarrow \mathrm{Hom}_k(V_1, V_2) : \varphi(k_1 g k_2) = k_1 \circ \varphi(g) \circ k_2, \ \mathrm{supp}\,\varphi \text{ cpt.}\}.$$

bimod. under conv.

$V_1 = V_2$: as before.

<u>As vector space:</u>

$$\exists\, \varphi \text{ with } \mathrm{supp}\,\varphi = K\lambda(\varpi)K, \quad \lambda \in Y(T)_-$$

$$\Leftrightarrow (V_1)_{N_\lambda(k)} \cong V_2^{N_{-\lambda}(k)}$$

$$\underset{\substack{\uparrow \\ \text{Lemma 3}}}{\Leftrightarrow} (V_1)_{N_\lambda(k)} \cong (V_2)_{N_\lambda(k)} \qquad (**)$$

Find: $\mathcal{H}_G(V_1, V_2) \neq 0 \Leftrightarrow (V_1)_{\bar{U}(k)} \cong (V_2)_{\bar{U}(k)}$.

$\Rightarrow \mathcal{H}_G(V_1) \underset{\text{Satake}}{\cong} \mathcal{H}_G(V_2)$

<u>Ex.</u>: $n = 2$, $V_1 = F(0,0) = \mathbb{1}$

$V_2 = F(q-1, 0)$ Steinberg, dim $= q$

$(**) \Leftrightarrow \lambda = (i,j)$ with $i < j$

Note: $F(q-1, 0)_{\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}} \cong F(q-1) \otimes F(0) = \mathbb{1}$

"minimal support": $i + 1 = j$.

Satake: $\mathcal{H}_G(V_1, V_2) \hookrightarrow \mathcal{H}_T((V_1)_{\bar{U}(k)}, (V_2)_{\bar{U}(k)})$.

compatible with compositions $\underbrace{\operatorname{ind} V_1 \to \operatorname{ind} V_2 \to \operatorname{ind} V_3}$.

$\Rightarrow$ If $\mathcal{H}_G(V_1, V_2) \neq 0$, $\underbrace{\mathcal{H}_G(V_1) \underset{\substack{\uparrow \\ \text{via Satake!}}}{\cong} \mathcal{H}_G(V_2)}_{\text{write } \mathcal{H}_G.}$ act in the same way.

So any non-zero map $\operatorname{ind}_K^G V_1 \to \operatorname{ind}_K^G V_2$ is $\mathcal{H}_G$-linear.

Given $\chi : \mathcal{H}_G \to \bar{k}$, get $\operatorname{ind}_K^G V_1 \otimes_{\mathcal{H}_G} \chi \to \operatorname{ind}_K^G V_2 \otimes_{\mathcal{H}_G} \chi$.

<u>Observation:</u> Say $\operatorname{ind} V_1 \underset{\theta_1^2}{\overset{\theta_2^1}{\rightleftarrows}} \operatorname{ind} V_2$.

Then $\theta_2^1 \circ \theta_1^2 = \theta_1^2 \circ \theta_2^1 \in \mathcal{H}_G$.

If $\chi(\text{\_\_\_\_\_\_\_\_}) \neq 0$, then $\operatorname{ind}_K^G V_1 \otimes_{\mathcal{H}_G} \chi \xrightarrow{\sim} \operatorname{ind}_K^G V_2 \otimes_{\mathcal{H}_G} \chi$.

<u>Prop. 2</u> (<u>Change Weight</u>)

$1 \leq i < n$.

$V_1 = F(\underline{a})$

$V_2 = F(\underline{b})$ with $b - a = (\underbrace{q-1, q-1, \ldots, q-1}_{i}, 0, \ldots, 0)$

($\Rightarrow a_i = a_{i+1}$!)

Suppose $\chi : \mathcal{H}_G \to \bar{k}$ is param^d by (M, χ_M), where the "Hecke Levi" M has a break at i:

$$M = \begin{pmatrix} \square & & & & \\ & \ddots & & & \\ & & \square\!\!\square & & \\ & & & \ddots & \\ & & & & \square \end{pmatrix}, \quad \text{Let } t_i := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \varpi & & & \\ & & & \varpi^{-1} & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$$\underbrace{}_{i}\underbrace{}_{n-i} \qquad\qquad \begin{array}{cc} \uparrow & \uparrow \\ i & i+1 \end{array}$$

(note: $\chi_M : Z_M \to \bar{k}^\times$)

Suppose $t_i \notin Z_M$ <u>or</u> $\chi_M(t_i) \neq 1$.

Then $\operatorname{ind}_K^G V_1 \otimes_{\mathcal{H}_G} \chi \cong \operatorname{ind}_K^G V_2 \otimes_{\mathcal{H}_G} \chi$.

<u>Rk:</u> If $t_i \in Z_M$ and $\chi_M(t_i) = 1$, then ~~the~~ two 1-blocks come together at i and the corresp. two components of χ_M are equal.

(ratio is unram. since $a_i = a_{i+1}$

<u>Idea of pf:</u>

take θ_2^1, θ_2^1 : minimal support $\quad \left[(0, 0, \dots, 0, \underbrace{1, \dots, 1}_{n-i}) \in Y(T) \right]$

Can compute $\theta_1^2 \circ \theta_2^1 \in \mathcal{H}_G$.

To compute its Satake transform, use explicit formula (Lusztig-Kato). //

<u>Generalised Steinbergs</u>

<u>Thm. 3</u> (Grosse-Klönne)

The irred. constit. of $\operatorname{Ind}_{\overline{B}}^G(1)$ are the gen^d Steinberg reps., each occurring with mult. 1:

$$\mathrm{Sp}_P = \frac{\operatorname{Ind}_{\overline{P}}^G(1)}{\sum\limits_{Q \supsetneq P} \operatorname{Ind}_{\overline{Q}}^G(1)}. \qquad \forall P \text{ std. parab.}$$

$\# = 2^{n-1}$

red: $K \longrightarrow G(k)$

<u>His proof:</u> Iwahori $\bar{I} = \mathrm{red}^{-1}(\bar{B}(k))$,
$\bar{I}(1) = \mathrm{red}^{-1}(\bar{U}(k))$.

Determine $\mathcal{S}_{pp}^{\bar{I}} = \mathcal{S}_{pp}^{\bar{I}(1)}$ and show that it is irred. as

$\bar{k}[\bar{I} \backslash G / \bar{I}]$-module.

Since $\bar{I}(1)$ is pro-p and $\mathcal{S}_{pp}^{\bar{I}}$ generates $\mathcal{S}_{pp}$ $\Rightarrow$ $\mathcal{S}_{pp}$ irred.

<u>Alternatively:</u> (all split red. gps)

Part of his proof $\Rightarrow$ $\mathcal{S}_{pp}$ contains a <u>unique</u> weight.

$\exists!$ weight V_p s.t. $(V_p)_{\bar{N}(k)} = \mathbb{1}$ and V_p is M-reg.

~~On Levi M: $\mathrm{ind}_{M(1)}^{M} \mathbb{1}$~~

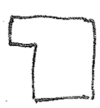
$(V_p)_{\bar{N}(k)} \cong \mathbb{1}$ is a weight of 1 (triv. rep. of M)

V_p - - - - - - - - $\mathrm{Ind}_{\bar{P}}^{G}(1)$. $\hookrightarrow \mathrm{Ind}_{\bar{B}}^{G}(1)$.

$\underset{\text{Lemma 5}}{\Longrightarrow}$ V_p generates $\mathrm{Ind}_{\bar{P}}^{G}(1)$. $\searrow$ $\mathcal{S}_{pp}$

$\underset{\text{irred. pf. (part I)}}{\Longrightarrow}$
V_p is M-reg

$\Longrightarrow$ $\boxed{\begin{array}{l} V_p \text{ is unique wt. of } \mathcal{S}_{pp}. \\ \text{Hecke evals.} \leftrightarrow (T, 1). \end{array}}$ $\Rightarrow$ $\mathcal{S}_{pp}$ irred.

<u>Ex.:</u> $p =$  $\begin{smallmatrix}1\\2\end{smallmatrix}$ $\Rightarrow$ $V_p \cong F(q-1, 0, 0)$.

Irreducibility of $\mathrm{Ind}_{\bar{P}}^{G}(\sigma_1 \otimes \cdots \otimes \sigma_r)$, part II

$M = \prod M_i$, $\quad M_i = GL_{n_i}(F)$

σ_i either supersing., $n_i \geq 2$

or $\sigma_i = \mathrm{Sp}_{Q_i} \otimes (\eta_i \circ \det)$, $\quad \eta_i : F^\times \to \bar{k}^\times$

$\boxed{\eta_i \neq \eta_{i+1}}$ $\forall i$

Recall: $0 \neq \pi \subset \mathrm{Ind}\, \sigma$.

Remains to show: π contains an M-reg. wt.

Say V is a wt. of π, with Hecke evals. χ.

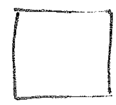
Say χ param^d by (L, χ_L): $\quad L$ std. Levi

$\chi_L : Z_L \to \bar{k}^\times$.

We can determine (L, χ_L):

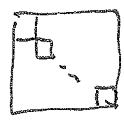
If σ_i ss., $n_i \geq 2$, all its Hecke evals. param^d by (M_i, ω_i) by def.

ω_i: c.c. of σ_i


≥ 2

If $\sigma_i = \mathrm{Sp}_{Q_i} \otimes (\eta_i \circ \det)$, - - - - - - - - - - $(T_i, \eta_i \circ \det)$

T_i: torus of M_i



Call this pair (M_i', χ_i').

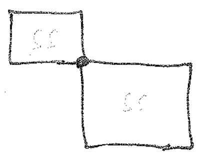
$\Rightarrow \sigma$: Hecke evals. $\mapsto (\prod M_i', \prod \chi_i')$.

$\Rightarrow \mathrm{Ind}\, \sigma$: - - - - - - - - - - - - -

Lemma 5

Now change weight:

① σ_i ss. $\forall i$
$n_i \geq 2$

$M =$ 

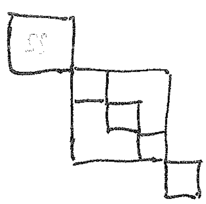
2
3

$V = F(a, \underline{\underline{b, c}}, d, e)$

$b > c$: M-reg $\Rightarrow$ done.

$b = c$: use Prop. 2
no consec. 1-blocks in Hecke Levi
$\Rightarrow$ can change to wt. $F(a+q-1, \underline{\underline{b+q-1, b}}, d, e)$
M-reg. ✓

② general case

$M =$ 

2
3
1

$V = F(a, \underline{\underline{b, c}}, d, \underline{\underline{e, f}})$

$b = c$: make it regular there as in prev. case

2 - block
1 - block

$e = f$: two consec. 1-blocks in Hecke Levi
can make it regular there since $\eta_2 \neq \eta_3$

□

Final goal: any irr. adm. π is of this form.

Ordinary parts (Emerton)

Fix $P = MN$ std. parab.

$$\mathrm{Ord}_P : \{\text{smooth } G\text{-reps.}\} \longrightarrow \{\text{smooth } M\text{-reps.}\} \quad \text{functor}$$

(even P-)

Facts:

(1) π adm. $\Rightarrow$ $\mathrm{Ord}_P\, \pi$ adm.

(2) π, σ adm.

$$\Rightarrow \mathrm{Hom}_G(\mathrm{Ind}_{\bar{P}}^G \sigma, \pi) \xrightarrow{\sim} \mathrm{Hom}_M(\sigma, \mathrm{Ord}_P\, \pi).$$

(3) π smooth   has central char.
σ locally Z_M-fin. (e.g. if ~~[illegible]~~)

$$\Rightarrow \mathrm{Hom}_G(\mathrm{Ind}_{\bar{P}}^G \sigma, \pi) \hookrightarrow \mathrm{Hom}_M(\sigma, \mathrm{Ord}_P\, \pi).$$

(4) σ adm.

$$\Rightarrow \mathrm{Ord}_P\, \mathrm{Ind}_{\bar{P}}^G \sigma \cong \sigma.$$

Rk: (2) is analogue of Bernstein's 2[nd] adjunction. (Jacquet mod.)

Lemma 6: If π is an irr. adm. G-rep. and $\mathrm{Ord}_P\, \pi \neq 0$, then $\exists$ σ irr. adm. M-rep. s.t. $\mathrm{Ind}_{\bar{P}}^G \sigma \twoheadrightarrow \pi$.

Pf: $\mathrm{Ord}_P\, \pi$ adm. $\Rightarrow$ $\exists\ \underset{\text{irr. adm.}}{\sigma} \hookrightarrow \mathrm{Ord}_P\, \pi$

$$\underset{\text{Fact (2)}}{\Longrightarrow} \mathrm{Ind}_{\bar{P}}^G \sigma \twoheadrightarrow \pi. \qquad \square$$

Def. of $\mathrm{Ord}_p \pi$:

$$M^+ := \{ m \in M : \; m N(\mathcal{O}) m^{-1} \subset N(\mathcal{O}) \}.$$

M^+ acts on $\pi^{N(\mathcal{O})}$:

$$m \in M^+, \quad v \in \pi^{N(\mathcal{O})}$$

$$m \bullet v := \sum_{N(\mathcal{O})/mN(\mathcal{O})m^{-1}} n m v$$

Then
$$\mathrm{Ord}_p \pi := \mathrm{Map}_{M^+}(M, \pi^{N(\mathcal{O})})_{Z_M\text{-fin}}$$
$$= \mathrm{Map}_{Z_M^+}(Z_M, \pi^{N(\mathcal{O})})_{Z_M\text{-fin}}. \qquad (Z_M^+ := M^+ \cap Z_M)$$

point: $M = M^+ \times_{Z_M^+} Z_M$

Classification, part I

π irr. adm.

π contains some wt. V with Hecke evals. $\chi \xleftrightarrow[\text{param.}]{} (M, \chi_M)$.

Recall (Lemma 4): M is smallest Levi s.t. χ fact. through $\mathcal{H}_M$.

Suppose $\exists$ std. parab. $Q = LN' \neq G$ s.t. V is L-reg. and χ fact. through $\mathcal{H}_L$ ($\Leftrightarrow M \subset L$).

$$\Rightarrow \quad \mathrm{ind}_K^G V \otimes_{\mathcal{H}_G} \chi \twoheadrightarrow \pi$$
$$\wr\wr$$
$$\mathrm{Ind}_{\bar{Q}}^G \left(\mathrm{ind}_{L(\mathcal{O})}^L V_{\bar{N}'(k)} \otimes_{\mathcal{H}_L} \chi \right) \qquad \text{by Prop. 1}$$

Fact (3) $\Rightarrow$ $\quad \mathrm{ind}\, V_{\bar{N}'(L)} \otimes \mathcal{H}_L \ \chi \xrightarrow[\neq 0]{} \mathrm{Ord}_Q \pi$

$\Rightarrow \quad \mathrm{Ord}_Q \pi \neq 0.$

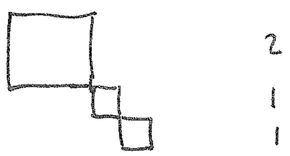
$\Rightarrow$ (Lemma 6) $\quad \exists\ \sigma$ irr. adm. s.t. $\mathrm{Ind}_{\bar{Q}}^{G} \sigma \twoheadrightarrow \pi$.

Write $\sigma = \sigma_1 \otimes \dots \otimes \sigma_r$ (Levi blocks of L).

Induct!

<u>How can find such a Levi L?</u>

e.g.

$M =$ 

$$\chi_M = \begin{pmatrix} \chi_1 & & \\ & \chi_2 & \\ & & \chi_3 \end{pmatrix}$$

$V = F(a, \underline{b, \overline{c}}, \overline{d})$

$b > c$: $\quad L = (2, 2)$-Levi works

$c > d$: $\quad L = (3, 1)$-Levi works

done in this case!

$b = c$: $\quad$ can change weight (not two consec. 1-blocks)

$c = d$: $\quad$ - - - - - - - - - - - <u>provided</u> $\chi_2 \neq \chi_3$.

$\Rightarrow$ the only cases when we <u>can't</u> find such a Levi L:

① $M = G$ (supersing.)

② $M = \begin{matrix}\square & & \\ & \ddots & \\ & & \square\end{matrix} = T \qquad \chi_M = \eta \circ \det$, some $\eta : F^\times \to \bar{k}^\times$.

$V = F(a, a, \dots, a)$

<u>Twist</u> $\Rightarrow \quad M = T, \quad \chi_M = 1.$
$\qquad\qquad V = \mathbb{1}$

looks like the triv. rep.! //

<u>Prop. 3</u>: If π irr. adm. contains the triv. wt. $\mathbb{1}$ with Hecke evals. $\leftrightarrow (T, 1)$,

then <u>either</u> $\mathrm{Ord}_P\, \pi \neq 0$, some $P \neq G$

<u>or</u> $\pi = 1$.

Classification, part II

By Prop. 3 and induction,

$\exists \ \mathrm{Ind}_{\bar{Q}}^{G} (\sigma_1 \otimes \dots \otimes \sigma_r) \twoheadrightarrow \pi$ where σ_i is ss. $\forall i$ or the triv. M_i-rep. (up to twist)

~~(includes char.!)~~

Now embed each triv. rep. into $\mathrm{Ind}^{M_i}(1)$.
So LHS $\hookrightarrow \mathrm{Ind}^{G}(\sigma_1' \otimes \dots \otimes \sigma_s')$ with σ_i' ss. $\forall i$ (this includes char.!)
Decompose ~~LHS~~ this representation:

e.g. $\quad \mathrm{Ind}_{\bar{Q}}^{G} (\sigma \otimes \chi \otimes \chi \otimes \eta \otimes \eta \otimes \eta) \qquad \sigma$ ss. of $GL_2(F)$
$\qquad\qquad \chi \neq \eta : F^\times \to \bar{k}^\times$.

$$\cong \mathrm{Ind}\,(\sigma \otimes \mathrm{Ind}^{GL_2}(\chi \otimes \chi) \otimes \mathrm{Ind}^{GL_3}(\eta \otimes \eta \otimes \eta)) \qquad (*)$$

Note: $\mathrm{Ind}^{GL_2}(\chi \otimes \chi) = \mathrm{Ind}^{GL_2}(1) \otimes (\chi \circ \det)$

By Thm. 3, the irr. constit. of $(*)$ are:

$$\mathrm{Ind}\left(\sigma \otimes \left(Sp_{Q_2} \otimes \chi \circ \det\right) \otimes \left(Sp_{Q_3} \otimes \eta \circ \det\right)\right) \qquad \chi \neq \eta$$

$\square$

Pf. of Prop. 3 $(n=2)$:

Pick Hecke evec. $v \in \pi^K = \mathrm{Hom}_K(\mathbb{1}, \pi)$ with evals $\leftrightarrow (T, 1)$.

$$U_1 := [I \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} I] \in \bar{k}[I \backslash G / I] \curvearrowright \pi^I$$

If U_1 has a non-0 eval. on π^I, then $\mathrm{Ord}_B \pi \neq 0$ ($\Rightarrow$ done).

(use: $\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} \in Z_T^+$ and $\pi^I \subset \pi^{N(\mathcal{O})}$)

non-0 evals $\Longrightarrow$ extend from Z_T^+ to Z_T.

So wlog. $\underline{U_1 \text{ is nilpotent on } \pi^I} \ni v$

Hecke evals. $\leftrightarrow (T, 1)$:

$$[K \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K] v = v.$$

$$\Leftrightarrow [I \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} I] v + [I \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix} I] v = v.$$

$$\Leftrightarrow \underbrace{S_1 \Pi v}_{U_1} + \Pi v = v. \qquad (*)$$

where $S_1 = [I \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} I]$, $\Pi = [I \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix} I]$

$S_1^2 = -S_1$ (quadratic)

$\Pi^2 = [I \begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} I] = 1$ on π^I

(central char.)

$S_1(*)$: $\quad \underline{\underline{0 = S_1 v}}$. (or cond. + self)

$U_1(*)$: $\quad U_1^2 v + \underbrace{U_1 \Pi v} = U_1 v.$

$\quad = S_1 \Pi^2 v = S_1 v = 0.$

$\therefore \underline{\underline{U_1 v}} = U_1^2 v = U_1^3 v = \ldots = \underline{\underline{0}} \qquad (\because U_1$ nilp.$)$

$(*) \Rightarrow \Pi v = v.$

$\Rightarrow v$ fixed by $\langle K, \Pi \rangle = G.$

$\Rightarrow \pi = 1. \qquad \square$

__Uniqueness:__

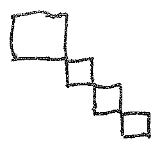
π irr. adm. $\Rightarrow \pi \cong \mathrm{Ind}_{\bar{P}}^{G}(\sigma_1 \otimes \ldots \otimes \sigma_r)$ as in Thm. 1

$\Rightarrow \pi$ has constant Hecke evals. (Ired., part II)

(L, χ_L)

$\Rightarrow$ can <u>recover inducing parab.</u> P

e.g. $L =$

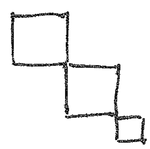


2
1
1
1

$\chi_L = \chi_1 \cdot \chi_2 \cdot \chi_2 \cdot \chi_3$

$\chi_1 \neq \chi_2, \chi_2 \neq \chi_3$

$\Rightarrow M =$



2
2
1

σ_1 : ss.

$\sigma_2 = Sp_{Q_2} \otimes (\chi_2 \circ \det) \qquad$ (same Q_2).

$\sigma_3 = \chi_3$

Fact (4) $\Rightarrow$ <u>recover σ.</u> $\square$

<u>supersing. = supercusp.:</u> π irr. adm.

By classification: • π supercusp $\Rightarrow$ π supersing.

• σ irr. adm. M-rep.

$\Rightarrow$ $\mathrm{Ind}_{\bar{P}}^{G}\sigma$ has finite length

and all constituents have same Hecke evals.

If π not supercusp.: π occurs in $\mathrm{Ind}_{\bar{P}}^{G}\sigma$, $P \neq G$.

$\Rightarrow$ Hecke evals. $\leftrightarrow$ $(M', \chi_{M'})$, some $M' \subset M \neq G$.

$\Rightarrow$ π not supersing. $\square$