MAT 347
Problems for Homework 18
March 13, 2019

1. Let K/M and M/F be field extensions. Prove or give a counterexample:

   (a) If K/F is normal, then K/M is normal.

   (b) If K/F is normal, then M/F is normal.

   (c) If K/M and M/F are normal, then K/F is normal.

2. Prove that every field extension of degree 2 is normal.

   Warning: Do not exclude characteristic 2!

   Note: Are you having a déjà vu? If these first two questions make you think of similar results for groups, there is a good reason for it.

3. The goal of this problem is to prove that a finite extension generated by separable elements is separable. (See Thm. 6.15(i) in the notes.)

   For the first few questions, let us fix a finite, normal field extension K/F. Given intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq K$, we define $\mathrm{Emb(M_2/M_1)}$ to be the set of M_1–homomorphisms $\varphi : M_2 \to K$. (Recall that this means that φ is a field homomorphism such that $\varphi|_{M_1} = \mathrm{id}$, the identity of M_1.)

   (a) Assume $M_2 = M_1(\alpha)$. Prove that $|\,\mathrm{Emb(M_2/M_1)}\,|$ equals the number of distinct roots of $m_{\alpha,M_1}(X)$ in K. Conclude that $|\,\mathrm{Emb(M_2/M_1)}\,| \leq [M_2 : M_1]$, with equality iff α is separable over M_1.

   (b) For any intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq K$, prove that

   $$|\,\mathrm{Emb(M_3/M_1)}\,| = |\,\mathrm{Emb(M_3/M_2)}\,|\,|\,\mathrm{Emb(M_2/M_1)}\,|.$$

   (*Hint:* One way to do this is to consider $\mathrm{Emb(M_2/M_1)} = \{\sigma_i\}$ and $\mathrm{Emb(M_3/M_2)} = \{\tau_j\}$. Try to extend each σ_i to an element $\widetilde{\sigma}_i \in \mathrm{Emb(K/M_1)}$ and then show that the $\widetilde{\sigma}_i \circ \tau_j$ are all the elements of $\mathrm{Emb(M_3/M_1)}$.)

   (c) Assume K/F is not separable. Prove that $|\,\mathrm{Emb(K/F)}\,| < [K : F]$.

   (d) Assume $K = F(\alpha_1, \ldots, \alpha_n)$, where each α_i is separable over F. Prove that $|\,\mathrm{Emb(K/F)}\,| = [K : F]$. Conclude that K/F is separable.

   For the remaining questions, we remove the initial assumptions.

(e) Prove that the splitting field of a separable polynomial is a separable extension. (*Hint:* try to use the previous parts.)

(f) Let K/F be a finite, separable extension. Prove that its normal closure is a finite, normal, separable extension.

(g) Let K/F be any finite extension (not necessarily normal). Assume that $K = F(\alpha_1, \ldots, \alpha_n)$ and that α_i is separable over F for all i. Prove that K/F is separable.

For practice (not collected)

4. Calculate the degree of $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[5]{2})/\mathbb{Q}$. What about $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \sqrt[5]{2})/\mathbb{Q}$? (*Hint:* try to use the tower law.)

5. Give an example of a finite field extension that is separable but not normal (resp. normal but not separable). What about a finite extension that is neither normal nor separable?

6. Suppose that $K = F(\alpha_1, \ldots, \alpha_n)$. Let L be a splitting field of
$$f(X) := m_{\alpha_1,F}(X) \cdots m_{\alpha_n,F}(X)$$
over K. Show that L is also a splitting field of $f(X)$ over F.

7. Suppose we have a tower of finite field extensions $L/N/K/F$, where N/F is a normal closure of K/F. Show that any F-homomorphism $\phi : K \to L$ satisfies $\phi(K) \subset N$.

8. Suppose that $L/K/F$ are finite field extensions. Prove that L/F is separable iff both L/K and K/F are separable. (*Hint:* for the difficult direction use Problem 3.)