UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2011 EXAMINATIONS
MAT327H1F – Topology

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Name: ______________________

Instructions: Solve 5 of the following 6 questions. Indicate clearly which question you do not want to count! (If you answer all six and do not give any indication, we will assume that you want questions 1–5 to count.) All questions carry equal weight though different parts of the same question may be weighted differently.

Duration: 3 hours

Aids: None

Justify your answers! Express yourself clearly and accurately!

Prob.	Possible points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
TOTAL	max. 50	

Question 1. *(10 points) (a) Suppose that X is a compact topological space and that $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a descending sequence of non-empty closed subsets of X. Prove that $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.*

(b) Define what it means for a metric space (X, d) to be bounded. Show that every compact metric space is bounded, and give an example of a bounded metric space that is not compact. Justify your answer.

Question 2. *(10 points) Recall that, as a set,* $\mathbb{R}^\omega = \prod_{n=1}^{\infty} \mathbb{R}$*. Let* $\mathbb{R}^\infty$ *denote the subset of* $\mathbb{R}^\omega$ *consisting of sequences that are "eventually zero", i.e., all sequences* $(x_1, x_2, \ldots)$ *such that* $x_n \neq 0$ *for only finitely many values of* n*. Determine the closure of* $\mathbb{R}^\infty$ *in* $\mathbb{R}^\omega$ *in the* <u>product</u> *and* <u>uniform</u> *topologies. Justify your answer.*

Question 3. *(10 points) Suppose that X is a Hausdorff topological space. Let $Y \subset X$ be a compact subspace and suppose that $x \notin Y$. Show that there exist disjoint open subsets U, V of X such that $x \in U$ and $Y \subset V$.*

Question 4. *(10 points) (a) Suppose that X is a connected topological space and that $f : X \to \mathbb{R}$ is a continuous function. Let x_1, $x_2 \in X$ and suppose that $c \in \mathbb{R}$ lies between $f(x_1)$ and $f(x_2)$. Show that there is an $x \in X$ such that $f(x) = c$. Quote in full any theorems you use.*

(b) Determine which of the following subspaces of $\mathbb{R}$ are homeomorphic: $(0, 1)$, $[0, 1)$, $(0, 1]$, $[0, 1]$. *Justify your answer.*

Question 5. *(10 points) We say that a subset* A *of a topological space* X *is* <u>*dense*</u> *in* X *if* $\overline{A} = X$*.*

(a) Show that A *is dense if and only if each non-empty open subset* U *of* X *intersects* A *(i.e.,* $U \cap A \neq \varnothing$*).*

(b) Suppose that A_1*,* A_2 *are two* <u>*open*</u> *dense subsets of* X*. Show that* $A_1 \cap A_2$ *is a dense subset of* X*. (Hint: use part (a).)*

(c) Give an example of a topological space X and subsets A_1, A_2 to show that $A_1 \cap A_2$ need not be dense if A_1 and A_2 are both dense.

Question 6. *(10 points) Recall that a G_δ-set in a topological space X is a subset A that is the intersection of a countable collection U_1, U_2, ... of open subsets.*

Suppose that X is a normal space. Show that there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$ if and only if A is a closed G_δ-set in X.

Quote in full any theorem you use.

Scratch Paper