BURNSIDE'S LEMMA
MAT 347

1 Definitions

Definitions. Let G be a group acting on a set A.

- Given $g \in G$, we define the *fixed set* of g as the set

$$\text{Fix}(g) := \{a \in A \mid g \cdot a = a\} \subseteq A$$

- Given $a \in A$, we define the *stabilizer* of a as the set

$$\text{Stab}(a) := \{g \in G \mid g \cdot a = a\} \subseteq G$$

- Given $a \in A$ we define the *orbit* of a as the set

$$\Omega_a := \{g \cdot a \mid g \in G\} \subseteq A$$

2 The results

The Orbit-Stabilizer Theorem. Let G be a group acting on a set A. Let $a \in A$. Then

$$|G| = |\Omega_a| \, |\text{Stab}(a)|$$

Proof. Given this fixed $a \in A$, we define a function $F : G \to \Omega_a$ by $F(g) := g \cdot a$ for all $g \in G$. By construction, F is surjective, but it won't be injective in general. Verify that this function is "k-to-1", where $k = |\text{Stab}(a)|$. In other words, every element $b \in \Omega_a$ has exactly k preimages. The result follows. □

Burnside's Lemma. Let G be a finite group acting on a set A. Then

$$\text{number of orbits of this action} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

Proof. Consider the set $X := \{(g, a) \in G \times A \mid g \cdot a = a\}$ Computing the cardinality of X in two different ways, notice that

$$|X| = \sum_{g \in G} |\text{Fix}(g)| = \sum_{a \in A} |\text{Stab}(a)|$$

Apply the Orbit-Stabilizer Theorem, and a bit of cleverness, to get the result. □

3 Exercises

1. Nine students are having an a cappella dance party. Some of them are doing vocal percussion and the rest are singing. How many ways can we arrange the drummers (D) and singers (S) in a circle? We will ignore rotation (so DSDSSDSSD is the same as SDSSDSSDD), but respect mirror images (so DSDSSDSSD is different from DDSSDSSDS). Consider, in your count, all possible ratios of drummers to singers – there may be no drummers and nine singers, nine drummers and no singers, or any combination of the two.

   Solution: 60

2. How many different ways are there to colour the faces of a cube with blue, red, and green? How many different ways are there to colour the edges of a cube with blue, red, and green? (Consider two colourings the same if one can be obtained from the other via a rotation, since you would not be able to to distinguished two such coloured cubes if I handed you both.)

   Solution: 57 for the faces; 22815 for the edges.

3. How many different necklaces can we make with n stones, if we have a large supply of stones of k colours. Answer this first for $n = 6$ and $n = 5$, then try the general case.

4. We want to make binary icosahedral dice. That is, we want to write the number 0 or 1 on each one of the faces of an icosahedron. (We obviously may repeat numbers, we do not need to use both numbers, and we do not need to use the same amount of each number). How many different dice can we build?

   Solution: 17824.

5. Consider a regular hexagonal prism (that is, a prism whose base is a regular hexagon). It has twelve vertices. We want to colour each vertex black or white. We consider two colourings as the same if they can be obtained from each other via a rigid motion in $\mathbb{R}^3$. (Be careful: there are more than 6 rotations.) In how many different ways can we colour the vertices of the prism?

   Solution: 382.

6. How many topologically different simple graphs are there with 4 vertices? And with 5 vertices? (A *simple* graph is one without loops, and such that between every two distinct vertices there is at most one edge.)

   Solution: 11 with 4 vertices; 34 with 5 vertices.