MAT 1110 — Additional exercises

March 8, 2013

Ex. 1: Recall that a sequence $1 \to K \xrightarrow{\varphi} G \xrightarrow{\psi} H \to 1$ of alg. grps. is exact if
$$\begin{cases} \text{(i) it's set-theoretically exact} \\ \text{(ii) } 0 \to \operatorname{Lie} K \xrightarrow{d\varphi} \operatorname{Lie} G \xrightarrow{d\psi} \operatorname{Lie} H \to 0 \text{ is exact.} \end{cases}$$

(a) Show that φ is a closed immersion $\iff$ φ inj. and $d\varphi$ inj.

(b) - - - - ψ is separable $\iff$ ψ surj. and $d\psi$ surj.

(c) Deduce the sequence is exact $\iff$ $\begin{cases} \text{(i) as before} \\ \text{(ii') } \varphi \text{ is a closed immersion} \\ \quad\;\; \psi \text{ is separable.} \end{cases}$

(d) If char $k = 0$ show that (i) $\Rightarrow$ (ii').

Ex. 2: If $N \leq H \leq G$ are closed subgps. and $N \lhd G$, then the natural map $H/N \to G/N$ is a closed immersion (so we can think of H/N as closed subgroup of G/N) and we have a canonical iso.
$$(G/N)/(H/N) \cong G/H \text{ of homog. } G\text{-spaces.}$$

Ex. 3: Suppose $N, H \leq G$ are closed subgps such that H normalizes N. Show that HN is a closed subgroup of G and that we have a canonical iso. $HN/N \cong H/(H \cap N)$ of alg. gps. (Assume char $k = 0$)
Find a counterexample when char $k > 0$.

Ex. 4: Suppose $\varphi: G \to H$ is a morphism of alg. gps. If char $k = 0$ show that φ induces an isom. $G/\ker\varphi \xrightarrow{\sim} \operatorname{im}\varphi$. If char $k > 0$, find a counterexample

Ex. 5: Suppose H is a closed subgp. of an alg. gp. G. Show that if both H and G/H are connected, then G is connected. (Use e.g. Springer ex. 5.5.9 (1))

Ex. 6: Suppose $\varphi: G \to H$ is a morphism of alg. gps. If $H_1 \leq H_2 \leq H$ are closed subgps., show that we have a canonical iso. $\varphi^{-1}(H_2)/\varphi^{-1}(H_1) \xrightarrow{\sim} H_2/H_1$.
(Hint: show $\operatorname{Lie} \varphi^{-1}(H_i) = (d\varphi)^{-1}(\operatorname{Lie} H_i)$.)