Lattices, patching, and multiplicity one for the Iwahori (Lecture 8)

§1 Introduction

Recall (Lect. 1):

$\mathbb{F}/\mathbb{F}_p$ fin. (large)

$F/\mathbb{Q}$ tot. real, p inert in F [simplicity]

D/F quat. algebra, split at p and exactly one infinite place

$\bar{r}: G_F \longrightarrow GL_2(\mathbb{F})$ abs. irred. s.t.

$$\pi_p(\bar{r}) := \varinjlim_{V_p} \mathrm{Hom}_{G_F}(\bar{r}, H^1_{et}(X_{V_pV^p} \times_F \bar{F}, \mathbb{F})) \neq 0$$

$\circlearrowleft$ $GL_2(F_p)$ [adm. smooth]

Shi. curve $V_pV^p \subset (D \otimes_F \mathbb{A}_F^\infty)^\times$

(+ "optimal level" V^p + TW conditions)

Thm: If $\bar{r}|_{G_{F_p}}$ is (weakly) generic semisimple, then $GK\text{-}\dim(\pi_p(\bar{r})) = f$.

Notation:

$$\begin{array}{ccc} K := GL_2(\mathcal{O}_{F_p}) & \supset & I \\ \cup & & \cup \\ K_1 & \subset & I_1 \end{array}$$

$Z_1 = Z \cap K_1$

$W(\bar{r}) := \{$ Serre wts. σ : $\sigma \hookrightarrow \pi_p(\bar{r})|_K \}$ [up to norm^n]

$\pi := \pi_p(\bar{r})$

Pf (outline):

(1) Show

$$[\pi[\mathfrak{m}_K^2] : \sigma] = [\mathrm{soc}_K \pi : \sigma]$$

$\forall \sigma \hookrightarrow \mathrm{soc}_K \pi$ ($\Leftrightarrow \sigma \in W(\bar{r})$), where $\mathfrak{m}_K \triangleleft \mathbb{F}[\![K_1/Z_1]\!]$ max. ideal.

② Deduce

$$[\pi[m_I^3] : \chi] = [\mathrm{soc}_I \pi : \chi]$$

$\forall \chi \hookrightarrow \mathrm{soc}_I \pi = \pi^{I_1}$, where

$m_I \triangleleft \mathbb{F}[[I_1/Z_1]]$ max. ideal.

③ Deduce GK-dim$(\pi) = f$.

Rk: • "optimal level" $\Rightarrow$ $[\mathrm{soc}_K \pi : \sigma] \leq 1$ $\forall \sigma$

[our result more general]

• We'll discuss ①, ② is easier,

③ is in Lecture 6.

• ① generalises LMS / HW who showed

$$\pi[m_K] \cong D_0(\bar{r})$$
$$\overset{\shortparallel}{\pi^{K_1}}$$

$$\Rightarrow [\pi[m_K] : \sigma] = 1 \quad \forall \sigma \in W(\bar{r})$$

In fact, we determine $\pi[m_K^2]$.

§2 Strategy for ①

Fix $\sigma \in W(\bar{r})$.

Need:

$$1 \overset{!}{=} [\pi[m_K^2] : \sigma]$$

$$= \dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Proj}_K \sigma, \pi[m_K^2])$$

[proj. env. for cpt. K/Z_1-mod.]

$$= \dim_{\mathbb{F}} \mathrm{Hom}_K(\mathrm{Proj}_K \sigma / m_K^2, \pi)$$

Patching functor (EGS):

$$M_\infty : \left\{ \begin{matrix} \text{cts. } K\text{-reps. on f.g.} \\ W(\mathbb{F})\text{-mod. } (+\text{c.c.}) \end{matrix} \right\} \longrightarrow \{\text{f.g. } R_\infty\text{-mod.}\}$$

exact, where

$$R_\infty = R_p[[\underline{X}]]$$

$m_\infty \triangleleft R_\infty$; R_p: unrestricted framed defo. ring of $\bar{r}|_{G_{F_p}}$

and

- $M_\infty(V)/m_\infty \cong \mathrm{Hom}_K(V,\pi)^\vee$

  $(\Rightarrow M_\infty(\sigma') \neq 0 \Leftrightarrow \sigma' \in W(\bar{r}))$

- "$M_\infty(V(\lambda)^\circ \otimes_{W(\mathbb{F})} \sigma(\tau)^\circ)$ is maximal CM over $R_p^{(\lambda,\tau)}[[\underline{X}]]$"

  alg. rep. — ILL — Hodge type (ρ-shifted)

$\Rightarrow$ ETS:

$$1 \overset{!}{=} \dim_{\mathbb{F}} M_\infty(\mathrm{Proj}_K \sigma / m_K^2)/m_\infty$$

$\Leftrightarrow$ (NAK) $M_\infty(\mathrm{Proj}_K \sigma / m_K^2)$ is a [nonzero] cyclic R_∞-mod.

<u>Steps:</u>

(a) $\exists$ cts. K-rep R on a f.g. free $W(\mathbb{F})$-mod. s.t. $R/p \cong \mathrm{Proj}_K \sigma / m_K^2$.

(b) Prove $M_\infty(R)$ is cyclic

$(\Rightarrow M_\infty(R)/p \cong M_\infty(\mathrm{Proj}_K \sigma/m_K^2)$ cyclic.)

§3 The lattice R

$L := F_p \ (\cong \mathbb{Q}_{p^f})$

$k := \mathcal{O}_{F_p}/(p)$

$\Gamma := GL_2(k) \ (= K/K_1)$

$P_\sigma := \mathrm{Proj}_\Gamma \sigma \ (= \mathrm{Proj}_K \sigma / m_K)$

$$\Rightarrow 0 \to \underbrace{m_K/m_K^2}_{sl_{2,k} \otimes \mathbb{F} \ [\text{adjoint action}]} \otimes_{\mathbb{F}} P_\sigma \to \mathrm{Proj}_K \sigma / m_K^2 \to P_\sigma \to 0$$

<u>Idea:</u>

- lift P_σ to $\tilde{P}_\sigma :=$ proj. env. of σ as $W(\mathbb{F})[\Gamma]$-mo

- $R_1 := \tilde{P}_\sigma$

  $R_2 := sl_{2,\mathcal{O}_L} \otimes_{\mathbb{Z}_p} \tilde{P}_\sigma = \bigoplus_{j=0}^{f-1} \underbrace{sl_{2,\mathcal{O}_L} \otimes_{\mathcal{O}_L, j} \tilde{P}_\sigma}_{=: R_{2,j}}$

  [jth emb. $\mathcal{O}_L \hookrightarrow W(\mathbb{F})$]

- try to find R inside $R_1 \oplus R_2$ [modification]

Lemma: $\exists$ non-can. isom. of Γ-reps.

$$R_{2,j}/p \cong P_\sigma \oplus P_{\sigma_{j+}} \oplus P_{\sigma_{j-}}.$$

(where $\sigma = (r_0, r_1, \dots)$
$\sigma_{j\pm} = (r_0, r_1, \dots, r_j \pm 2, \dots)$)

$\rightsquigarrow$ choose $\iota_j : R_1/p = P_\sigma \hookrightarrow R_{2,j}/p$

$\Rightarrow$ diagonal $\iota : R_1/p = P_\sigma \hookrightarrow \bigoplus_j R_{2,j}/p = R_2/p$

Def.: $R := R_1 \times_{\substack{R_2/p \\ \iota}} R_2 \subset R_1 \oplus R_2$ [K-stable lattice]

Rk: Alternatively,

$$R = R_1 \times_{P_\sigma} R'_{2,0} \times_{P_\sigma} \cdots \times_{P_\sigma} R'_{2,f-1},$$

where

$$R'_{2,j} := P_\sigma \times_{\substack{R_{2,j}/p \\ \iota_j}} R_{2,j} \hookrightarrow R_{2,j}.$$

Prop.: (a) $0 \to R_2/p \to R/p \to P_\sigma \to 0$

(b) $(R/p)_{K_1} \cong P_\sigma$

(c) $R/p \cong \mathrm{Proj}_K \sigma / \mathfrak{m}_K^2$.

Pf: (a) From def^n:

$$0 \to R_2/p \to R/p \to \underset{}{\overset{R_1/p}{\overset{\parallel}{P_\sigma}}} \to 0 \quad \text{exact.}$$

$r_2 \mapsto (0, pr_2)$

$(r_1, r_2) \mapsto \bar{r}_1$

(b) As K_1 trivial on R_i/p,

$$\begin{tikzcd} H_1(K_1/Z_1, P_\sigma) \arrow[r, "\delta"] \arrow[d, equal] & R_2/p \arrow[r] \arrow[d, equal] & (R/p)_{K_1} \arrow[r] & P_\sigma \arrow[r] & 0 \\ sl_{2,k} \otimes P_\sigma & sl_{2,k} \otimes P_\sigma & & & \end{tikzcd}$$

(Γ-reps)

ETS: δ surj. ($\Leftrightarrow$ inj.)

Calculate:

$$\delta : sl_{2,k} \otimes P_\sigma \xrightarrow{id \otimes \eta} sl_{2,k} \otimes_k sl_{2,k} \otimes P_\sigma \xrightarrow{[\,,\,] \otimes id} sl_{2,k} \otimes P_\sigma$$

ETS: δ is inj. on Γ-socle.

$$\begin{array}{ccccc} \cup & & \cup & & \cup \\ sl_{2,k} \otimes \sigma & \longrightarrow & sl_{2,k} \otimes_k sl_{2,k} \otimes \sigma & \longrightarrow & sl_{2,k} \otimes \sigma \end{array}$$

[socle]

Lie alg. calculation ... [split by embedding $k \hookrightarrow \mathbb{F}$]

(c) By (b),

$$\underbrace{\mathrm{cosoc}_K(R/p)}_{K_1 \text{ triv.!}} \cong \mathrm{cosoc}_K(P_\sigma) \cong \sigma$$

$$\rightsquigarrow \begin{tikzcd} \mathrm{Proj}_K \sigma \arrow[r, two heads] \arrow[d] & R/p \\ \mathrm{Proj}_K \sigma / m_K^2 \arrow[ur, dashed, two heads, "\text{by (a)}"'] & \end{tikzcd}$$

isom. by dim. reasons. $\square$

§4 Cyclicity results

Recall: $R = \tilde{P}_\sigma \times_{P_\sigma} R'_{2,0} \times_{P_\sigma} \cdots \times_{P_\sigma} R'_{2,f-1}$.

Thm: $M_\infty(R)$ cyclic.

For simplicity, show $M_\infty(\tilde{P}_\sigma \times_{P_\sigma} R'_{2,j})$ cyclic

$$\| $$

$$M_\infty(\tilde{P}_\sigma) \times_{M_\infty(P_\sigma)} M_\infty(R'_{2,j})$$

(then induct).

Lemma 1 (EGS): If $M'' \subsetneq M' \subset M$ over local ring,

M' and M/M'' cyclic $\Rightarrow$ M cyclic.

Lemma 2 (EGS): If $M_1 \twoheadrightarrow M_3 \leftarrow M_2$ cyclic,

and $\mathrm{Ann}\, M_1 + \mathrm{Ann}\, M_2 = \mathrm{Ann}\, M_3$,

then $M_1 \times_{M_3} M_2$ cyclic.

Prop (LMS / HW): $M_\infty(\tilde{P}_\sigma)$ cyclic.

$\left(\underset{\text{NAK}}{\iff} M_\infty(P_\sigma) \text{ cyclic.} \right)$

Prop.: $M_\infty(R'_{2,j})$ cyclic.

Pf:

$$\begin{array}{ccccccccc} & & & & R/p & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & P_{\sigma_{j+}} \oplus P_{\sigma_{j-}} & \to & R'_{2,j}/p & \to & P_\sigma & \to & 0 \end{array}$$

$\rightsquigarrow \quad W := (R/p) / \sum_j z_j(P_\sigma) \twoheadrightarrow R'_{2,j}/p$

ETS: $M_\infty(W)$ is cyclic.

Prove $M_\infty(Q)$ cyclic $\forall\, W \twoheadrightarrow Q$ by ind^n on $\ell(Q)$

- $\ell(Q) = 1 \Rightarrow Q \cong \sigma$, done by "Diamond"

(EGS = Emerton – Gee – Savitt
LMS = Le – Morra – Schraen
HW = Hu – Wang)

- Key case:

$$\mathrm{soc}_K Q = \sigma' \in W(\bar{F}),\ \sigma' \not\cong \sigma \text{ and } [Q:\sigma] > 1.$$

$$\begin{tikzcd} & R/p \arrow[d, phantom, "\|"] & & \\ \Rightarrow \exists & \mathrm{Proj}_K \sigma / \mathfrak{m}_K^2 \arrow[rr] \arrow[dd, two heads] \arrow[ddrr, two heads] & & Q \text{ nonzero, not surj.} \\ & & & \cup\!\neq \\ & W \arrow[rr, dashed, two heads, "\text{Fact! (rep. theory)}"'] & & Q' \end{tikzcd}$$

Then $M_\infty(\sigma') \subsetneq M_\infty(Q') \subset M_\infty(Q)$.

Lemma 1 + induction $\Rightarrow$ $M_\infty(Q)$ cyclic. $\square$

To show $M_\infty(\tilde{P}_\sigma) \times_{M_\infty(P_\sigma)} M_\infty(R'_{2,j})$ cyclic, use

Lemma 2. Need $\mathrm{Ann}_{R_\infty} M_\infty(\dots)$.

Recall: $R_\infty = R_p[[\underline{X}]]$.

For (λ, τ) a Hodge type, let

$$\mathfrak{q}_{\lambda,\tau} := \ker(R_\infty \twoheadrightarrow R_p^{(\lambda,\tau)}[[\underline{X}]]).$$

Let

$$T := \{\text{tame inertial types } \tau : \sigma \in JH(\overline{\sigma(\tau)})\},$$

where $|T| = 2^f$.

<u>Lemma:</u>

(i) $\mathrm{Ann}_{R_\infty} M_\infty(\tilde{P}_\sigma) = \bigcap_{\tau \in T} \mathfrak{q}_{(1,0),\tau}$

(ii) $\mathrm{Ann}_{R_\infty} M_\infty(R'_{2,j}) = \bigcap_{\tau \in T} \mathfrak{q}_{\lambda,\tau}$, where

$$\lambda_\ell = \begin{cases} (1,0), & \ell \neq j \\ (2,-1), & \ell = j \end{cases}$$

(Use: $\tilde{P}_\sigma[1/p] \cong \bigoplus_{\tau \in T} \sigma(\tau)$ and "max. CM")

Let $\overline{R}_\infty := R_\infty / \bigcap \mathfrak{q}_{\lambda,\tau}$ over the 4^f

Hodge types (λ, τ) s.t. $\lambda_\ell \in \{(1,0),(2,-1)\}$, $\tau \in T$.

Prop. (cf. Lect. 7):

(i) $\overline{R}_\infty \cong \left(\widehat{\bigotimes}_{0 \le \ell \le f-1} S^{(\ell)} \right) [\![\underline{Y}]\!]$

(ii) $S^{(\ell)}$ has 4 min. primes $\mathfrak{p}_\ell^{\lambda_\ell, i_\ell}$,

where $\lambda_\ell \in \{(1,0),(2,-1)\}$, $i_\ell \in \{1,2\}$.

(iii) $\exists$ bijection $\theta : T \xrightarrow{\sim} \{1,2\}^f$ s.t.

under isom. in (i):

$$\mathfrak{q}_{\lambda,\tau} = \sum_\ell \mathfrak{p}_\ell^{\lambda_\ell, \theta(\tau)_\ell} \lhd \overline{R}_\infty.$$

Let $I_\ell^{\lambda_\ell} := \mathfrak{p}_\ell^{\lambda_\ell, 1} \cap \mathfrak{p}_\ell^{\lambda_\ell, 2}$.

Deduce using Lemma,

$$\operatorname{Ann} M_\infty(\tilde{P}_\sigma) = \bigcap_i \Big(\sum_\ell \mathfrak{p}_\ell^{(1,0), i_\ell} \Big) = \sum_\ell I_\ell^{(1,0)} \quad (1)$$

$$\operatorname{Ann} M_\infty(R'_{2,j}) = \Big(\sum_{\ell \neq j} I_\ell^{(1,0)} \Big) + I_j^{(2,-1)} \quad (2)$$

$$\operatorname{Ann} M_\infty(P_\sigma) = (p) + \sum_\ell I_\ell^{(1,0)}. \quad (3)$$

For Lemma 2 need

$$(3) \stackrel{!}{=} (1)+(2) \quad \Leftarrow \quad \underbrace{p \in I_j^{(1,0)} + I_j^{(2,-1)}}_{\text{Lecture 7!}}$$

$\square$