Lecture 6
Self-duality, finiteness results in semisimple case

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Essen Spring School
April 28, 2021

Notation. Keep (mostly) the notation in previous lectures.

- K = unramified extension over $\mathbb{Q}_p$ of degree f ;
- $\mathcal{O}$ = integers of K, $\mathbb{F}_q \cong \mathcal{O}_K/p$;
- $G = \mathrm{GL}_2(K)$, Z =center ;
- $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $\overline{B} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$;
- I = Iwahori, I_1 = pro-p-Iwahori, $H := \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix} \cong I/I_1$;
- $K_1 = \mathrm{Ker}\big(\mathrm{GL}_2(\mathcal{O}_K) \to \mathrm{GL}_2(\mathbb{F}_q)\big)$, $Z_1 = Z \cap K_1$;
- $(E, \mathcal{O}, \mathbb{F})$ = rings of coefficients.

$$\begin{pmatrix} \omega_{2f}^{(\Sigma r_i+1)p^i} & \\ & \omega_{2f}^{(\text{same})pf} \end{pmatrix}$$

Fix $\bar{\rho} : G_K := \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_2(\mathbb{F})$ cont., written in usual form with genericity conditions (slightly modified) :

- **generic** : if $10 \leq r_i \leq p - 12$ (as in [BHHMS1]) ;
- **strongly generic** : $\max\{10, 2f\} \leq r_i \leq p - \max\{12, 2f + 2\}$ (as in [BHHMS2]).

Let $\pi_v(\bar{r})$ = smooth admissible representation of G corresponding to some globalization $\bar{r}$ of $\bar{\rho}$ in mod p cohomology (cf. Lecture 1).

Start with :

Fact : if π is an irreducible admissible $\mathbb{C}$-representation of G, then

$$\hat{\pi} \cong \pi \otimes (\zeta^{-1} \circ \det),$$

$\pi^{\vee}$

where $\hat{\pi} := \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})^{\infty}$ denotes the *contragredient* (or smooth dual) of π, and $\zeta =$ the central character of π.

<u>Galois side</u> : 2-dimensional representation is dual to itself up to twist :

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, then

$${}^{T}g^{-1} = \det(g)^{-1} \cdot (w^{-1} g w).$$

$$\rho^{\vee} \simeq \rho \otimes (\det \rho)^{-1}$$

$\pi(\bar{\rho})/_{\mathbb{F}}$ self-duality

However, if π is over $\mathbb{F}$, $\hat{\pi}$ is usually zero (Livné, Vignéras) !

Set $\Lambda := \mathbb{F}[\![P]\!]$, where $P =$ pro-p open subgroup of G.
For finitely generated Λ-module M, set (cf. Lecture 1)

$$\mathrm{E}^i(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda).$$

Recall : Λ has global dimension $4f$.

Define

$$\text{grade} \quad j_\Lambda(M) := \min\{i \geq 0 : \mathrm{E}^i(M) \neq 0\},$$

can.

$$\text{dimension} \quad \delta_\Lambda(M) := \mathrm{gld}(\Lambda) - j_\Lambda(M).$$

Say M is **Cohen-Macaulay** if $j_\Lambda(M) = \mathrm{pd}_\Lambda(M)$. (e.g. projective $\Rightarrow$ CM)

Remark (Venjakob) :

- $\mathrm{pd}_\Lambda(M) = \max\{i \geq 0 : \mathrm{E}^i(M) \neq 0\}$.
- Λ satisfies **Auslander condition** : for any $N \subset \mathrm{E}^j(M)$, $j_\Lambda(N) \geq j$.

dim $\leq$ gld $-j$

Now consider $\mathfrak{C}_G :=$ category of f.g. (left) Λ-modules together with a compatible action of G. (**Example** : $\pi^\vee \in \mathfrak{C}_G$ for admissible π).
Then $\mathrm{E}^i(M) \in \mathfrak{C}_G$. $\circlearrowleft G$ [Kohlhaase]

Definition

Let $M \in \mathfrak{C}_G$ be Cohen-Macaulay. We say M is *essentially self-dual*, if

$$\mathrm{E}^{j_\Lambda(M)}(M) \cong M \otimes (\zeta \circ \det)$$

for some ζ.

Example. Below $i = 3f$.

(a) (Kohlhaase) $\mathrm{E}^i((\mathrm{Ind}_B^G \chi)^\vee) \cong (\mathrm{Ind}_B^G \chi^{-1}\alpha_B)^\vee$, where $\alpha_B := \omega \otimes \omega^{-1}$. Hence

$$\mathrm{E}^i\big((\mathrm{Ind}_B^G \underbrace{\chi_1\omega^{-1} \otimes \chi_2}_{\pi_0})^\vee\big) \cong (\mathrm{Ind}_B^G \underbrace{\chi_2\omega^{-1} \otimes \chi_1}_{\pi_1})^\vee \otimes (\zeta \circ \det).$$

(b) (Kohlhaase) $K = \mathbb{Q}_p$, π supersingular, then

$$\mathrm{E}^i(\pi^\vee) \cong \pi^\vee \otimes (\zeta \circ \det).$$

✓ Breuil

$f=1$, $\pi_v(\bar{r}) = \pi(\bar{\rho})^{\oplus r}$, where $\pi(\bar{\rho})$ — s.s. / $\pi_0 \oplus \pi_1$ / $\pi_0 - \pi_1$

(c) Let $\widetilde{H}^0$ be the space of *mod p* modular forms of level U^v in global setting (ii) or (iii) of Lecture 1 (or $\widetilde{H}^1$ in setting (i)).

← homology $\mathfrak{m} \leftrightarrow \mathfrak{m}_{\bar{r}} \subseteq \mathbb{T}$

Theorem (Calegari-Emerton, Hill). $(\widetilde{H}^0_{\mathfrak{m}})^\vee$ is projective (hence Cohen-Macaulay), and have $\mathbb{T} \times G$-equivariant isomorphism

$$\mathrm{E}^0((\widetilde{H}^0_{\mathfrak{m}})^\vee) \cong (\widetilde{H}^0_{\mathfrak{m}})^\vee \otimes \zeta.$$

Theorem 1

If $\mathrm{GK}(\pi_v(\bar{r})) \leq f$, then $\pi_v(\bar{r})^\vee$ is essentially self-dual.

Recall (Lecture 1) : $\delta_\Lambda(\pi_v(\bar{r})^\vee)$ is denoted $\mathrm{GK}(\pi_v(\bar{r}))$.

Patching module

univ def ring of $\bar{\rho}$

Let $R_\infty = R_{\bar{\rho}}^{\square} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![x_1, \dots, x_r]\!]$. Following [CEGGPS], a patching module is a non-zero $R_\infty[G]$-module $\mathbb{M}_\infty$ satisfying (among others) :

- $\mathbb{M}_\infty / \mathfrak{m}_\infty \cong \pi_v(\bar{r})^\vee$; ($\subset R_\infty$)
- $\mathbb{M}_\infty$ is a finitely generated $R_\infty[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-module ;
- $\exists$ regular local ring S_∞ (together with $S_\infty \to R_\infty$), such that $\mathbb{M}_\infty$ is f.g. projective $S_\infty[\![\mathrm{GL}_2(\mathcal{O}_K)]\!]$-module. Moreover,

$$\mathbb{M}_\infty \otimes_{S_\infty} \mathbb{F} \cong (\widetilde{H}^0_{\mathfrak{m}})^\vee . \quad /\mathbb{F}$$

Theorem 1 follows from Example (c) and :

Theorem (Miracle flatness, [GN])

Assume $R_{\bar{\rho}}^{\square}$ is **regular**. If

$$\mathrm{GK}(\pi_v(\bar{r})) \leq f,$$

then the equality holds, $\mathbb{M}_\infty$ is flat over R_∞, and $R_\infty \otimes_{S_\infty} \mathcal{O} \cong \mathbb{T}_{\mathfrak{m}}$ is complete intersection.

(**Caution** : It is crucial that $R_{\bar{\rho}}^{\square}$ is regular.)

$\Rightarrow$ $(\tilde{H}^0_{\mathfrak{m}})^\vee$ is flat over $\mathbb{T}_{\mathfrak{m}}/\mathfrak{m}$. $\Rightarrow$ $\pi_v(\bar{r})^\vee$ is self-dual.

ess. self-dual c. I.

$\subseteq G/Z_1$

$\mathfrak{gl}d = 3f.$

Notation. Write $\Lambda = \mathbb{F}[\![I_1/Z_1]\!]$ from now on.

$GK(\pi(\bar{r})) = f \Rightarrow j = 2f$

Recall (Lecture 5) : $\mathrm{gr}_{\mathfrak{m}_{I_1}}(\Lambda)$ is isomorphic to

$$\bigotimes_{i=0}^{f-1} \mathbb{F}[y_i, z_i, h_i]$$

where $[y_i, z_i] = h_i$, $[h_i, y_i] = [h_i, z_i] = 0$, and variables with $i \neq j$ commute. Moreover, $\deg(y_i) = \deg(z_i) = 1$.

The action of $g \in H := \begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}$:

$$g \cdot y_i = \alpha(g)^{p^i} y_i, \quad g \cdot z_i = \alpha(g)^{-p^i} z_i, \Rightarrow g \cdot h_i = h_i$$

where α sends $\begin{pmatrix} [a] & 0 \\ 0 & [d] \end{pmatrix}$ to ad^{-1} (via fixed embedding $\mathbb{F}_q \hookrightarrow \mathbb{F}$).

$f=1$. $y = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$, $\quad \begin{pmatrix} a & \\ & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ad^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & \\ & d \end{pmatrix}$

Let

$\ni h_i$

$$J := (y_i z_i, z_i y_i,\ 0 \le i \le f-1).$$

Lemma (cf. Lecture 5)

$\mathrm{gr}(\Lambda)/J$ is isomorphic to the commutative ring

$$\mathbb{F}[y_i, z_i; 0 \le i \le f-1]/(y_i z_i; 0 \le i \le f-1).$$

$\leftarrow$ Krull dim f

$\leftarrow$ 2^f min. prime ideals.

- If N is a f.g. $\mathrm{gr}(\Lambda)$-module killed by a power of J, can define $m_{\mathfrak{p}}(N)$, where $\mathfrak{p}$ is a minimal prime ideal of $\mathrm{gr}(\Lambda)/J$. Let

$$\mathfrak{p}_0 := (z_0, \dots, z_{f-1}).$$

- Let $\mathcal{C}_1$ be the category of smooth adm. π (with central character) such that $\mathrm{gr}(\pi^\vee)$ is killed by a power of J (cf. Lecture 5). This is an abelian category and stable under extensions and $\mathrm{E}^i_\Lambda(-)$.

Main result

Write $m_{\mathfrak{p}}(\pi)$ for $m_{\mathfrak{p}}(\mathrm{gr}(\pi^{\vee}))$ and $\mathbb{V} = \mathbb{V}_{\mathrm{GL}_2}$.

Theorem (cf. Lecture 5)

For $\pi \in \mathcal{C}_1$, $\dim_{\mathbb{F}} \mathbb{V}(\pi) \leq m_{\mathfrak{p}_0}(\pi)$.

Theorem 2 (BHHMS)

Let $\bar{\rho}$ be semisimple and generic. Then $\pi_v(\bar{r}) \in \mathcal{C}_1$ and $m_{\mathfrak{p}_0}(\pi_v(\bar{r})) \leq 2^f r$.

Together with the lower bound in Lecture 4, we deduce

Corollary 3 (BHHMS2)

If $\bar{\rho}$ is semisimple and strongly generic, then $\mathbb{V}(\pi_v(\bar{r})) \cong (\mathrm{ind}_K^{\otimes \mathbb{Q}_p} \bar{\rho})^{\oplus r}$.

Assume $r = 1$ (for simplicity).

Key ingredient : "multiplicity free property" ($r = 1$)

H

$$[\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3] : \chi] = 1, \quad \forall \chi \in \pi_v(\bar{r})^{I_1}. = \mathrm{soc}_I\, \pi_v(\bar{r}).$$

This will be proved in Lecture 7,8.

Remark. This multiplicity-freeness implies immediately $\mathrm{GK}(\pi_v(\bar{r})) \leq f$.

If $\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3]$ is multiplicity free, then $y_i z_i$ and $z_i y_i$ act trivially on $\mathrm{gr}^0(\pi_v(\bar{r})^\vee)$. Thus $\mathrm{gr}(\pi_v(\bar{r})^\vee)$ is finitely generated module over $\mathrm{gr}(\Lambda)/J \cong \mathbb{F}[y_i, z_i]/(y_i z_i)$, which has dimension f. □

$\mathrm{gr}(\pi(\bar{r})^\vee)$ is generated by $\mathrm{gr}^0(\pi(\bar{r})^\vee)$.

$e \in$

gr^0 ; e, of char χ

$\Rightarrow \pi_v(\bar{r}) \in \mathcal{C}_1$

gr^{-2} $y_i z_i e$. $z_i y_i e$. $\to$ char χ

$\mathrm{gr}(-)$ is killed by J.

multi. free $\Rightarrow y_i z_i e = z_i y_i e = 0.$

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Example. For $f = 1$, $\mathrm{gr}^{\geq -2}(F[\![I_1/Z_1]\!])$ looks like

$$\begin{array}{lccccccc}
\mathrm{gr}^0 & & & & 1 & & & \\
\mathrm{gr}^{-1} & & y\ \alpha & & \oplus & & \alpha^{-1}\ z & \\
\mathrm{gr}^{-2} & \underset{y^2}{\alpha^2} & \oplus & \underset{zy}{1} & \oplus & \underset{yz}{1} & \oplus & \underset{z^2}{\alpha^{-2}}
\end{array}$$

$\Rightarrow \mathbb{F}[y,z]/(y\cdot z)$.

The proof of Theorem 2

Upshot Construct an explicit $\mathrm{gr}(\Lambda)$-module N with $\underline{m_{\mathfrak{p}_0}(N) = 2^f}$, s.t.

upper bound $\dim V(\pi(\bar{r})) \leq m_{\mathfrak{p}_0}(\pi(\bar{r}))$

$$N \twoheadrightarrow \mathrm{gr}(\pi_v(\bar{r})^\vee).$$

$\Rightarrow m_{\mathfrak{p}_0}(\pi(\bar{r})) \leq 2^f$

(An "obvious" such module is

$$N' := \bigoplus_{\chi \in \pi_v(\bar{r})^{I_1}} \underline{(\mathrm{gr}(\Lambda)/J)} \otimes \chi^\vee. \Rightarrow \underline{\mathrm{gr}(\pi(\bar{r})^\vee)} \text{ killed by } J$$

$\Rightarrow m_{\mathfrak{p}_0} = 1$

But $m_{\mathfrak{p}_0}(N') = \dim \pi_v(\bar{r})^{I_1}$, which (often) $> 2^f = |W(\bar{\rho})|$.)

Proof. divide $\pi_v(\bar{r})^{I_1} = P \cup P^c$. where $P = \{\chi = \sigma^{I_1},\ \sigma \in \mathrm{soc}\, \pi(\bar{r})\}$
$\in W(\bar{\rho})$.

define $N = \left[\bigoplus_{\chi \in P} (\mathrm{gr}(\Lambda)/\underline{\mathfrak{a}_\chi} \otimes \chi^\vee)\right] \oplus \left(\bigoplus_{\chi \in P^c} \cdots\right)$

where $\mathfrak{a}_\chi$ ideal of $\mathrm{gr}(\Lambda)$, containing J, determined by relation between χ.

$\underline{\text{point}}$: check if $\chi \in P^c$, $y_i \in \mathfrak{a}_\chi$ for some i. $\Rightarrow (\mathrm{gr}(\Lambda)/\mathfrak{a}_\chi)_{\mathfrak{p}_0} = 0$, $\mathfrak{p}_0 = (z_0, \cdots z_{f-1})$

$\Rightarrow m_{\mathfrak{p}_0}(-) = 0$

$\Rightarrow m_{\mathfrak{p}_0}(N) = m_{\mathfrak{p}_0}\left(\bigoplus_{\chi \in P} \cdots\right) \leq |P| = 2^f$

Example 1. $f = 1$.

$gr(\pi(\bar{r})^\vee) \simeq [gr(\Lambda)/J \otimes \chi^\vee] \oplus [gr(\Lambda)/J \otimes (\chi^s)^\vee]$

$\simeq \chi \oplus \chi^s = |W(\bar{\rho})|$

(a) If $\bar{\rho}$ is irreducible, then $\pi_v(\bar{r})$ is supersingular, and $\dim \pi_v(\bar{r})^{I_1} = 2$ (Breuil, cf. Lecture 2), then take $N = N'$ (Thm of Paškūnas).

(b) If $\bar{\rho}$ is reducible split, then $\pi_v(\bar{r}) = \pi_0 \oplus \pi_1$ (both PS), and

$\sigma_1 \circlearrowleft \qquad \sigma_1 \circlearrowleft$

$$\pi_v(\bar{r})^{I_1} = (\chi_{\sigma_0} \oplus \chi^s_{\sigma_0}) \oplus (\chi_{\sigma_1} \oplus \chi^s_{\sigma_1}). \qquad \text{4-dim}$$

Fact: $\chi^s_{\sigma_0} = \chi_{\sigma_1}\alpha$, $\chi^s_{\sigma_1} = \chi_{\sigma_0}\alpha$.

$gr^0(\pi(\bar{r})^\vee)$: has dual basis e_0, e_0', e_1, e_1'

$\chi_{\sigma_0}^\vee \quad (\chi^s_{\sigma_0})^\vee \quad \chi_{\sigma_1}^\vee \quad (\chi^s_{\sigma_1})^\vee$

e_0

$y e_0 \qquad z \cdot e_0 \leftarrow \chi_{\sigma_0}^\vee \cdot \alpha^{-1} = (\chi^s_{\sigma_1})^\vee$

multi free $\Rightarrow$ $z \cdot e_0 = 0$.
$y \cdot e_0' = 0$
$z \cdot e_1 = 0$
$y \cdot e_1' = 0$.

take $N = \left(gr(\Lambda)/\underline{(J, z)} \otimes \chi_{\sigma_0}^\vee \right) \oplus (gr(\Lambda)/(J, y)) \otimes (\chi^s_{\sigma_0})^\vee \oplus \cdots$ $\qquad (J,z) = \mathfrak{a}_x$

$\twoheadrightarrow gr(\pi(\bar{r})^\vee)$

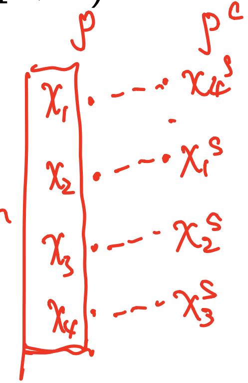
Example 2. $f = 2$, $\bar{\rho}$ irreducible. Then $W(\bar{\rho}) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where

$$\sigma_1 = (r_0, r_1), \quad \sigma_2 = (r_0 - 1, p - 2 - r_1)$$

$$\sigma_3 = (p - 1 - r_0, p - 3 - r_1), \quad \sigma_4 = (p - 2 - r_0, r_1 + 1)$$

(up to twist, cf. Lecture 3). Moreover,

$$\pi_v(\bar{r})^{I_1} \cong \bigoplus_{i=1}^{4} (\chi_{\sigma_i} \oplus \chi_{\sigma_i}^s). \quad = 8\text{-dim}$$



One checks $\chi_{\sigma_3}^s = \chi_{\sigma_1} \alpha^p$, etc.

By multiplicity freeness of $\pi_v(\bar{r})[\mathfrak{m}_{I_1}^3]$, take N to be

$$(\mathrm{gr}(\Lambda)/\underbrace{(J, z_1)}_{a_{\chi_1} \in \mathfrak{p}} \otimes \chi_{\sigma_1}^\vee) \oplus (\mathrm{gr}(\Lambda)/\underbrace{(J, y_1)}_{y_i^c} \otimes \underbrace{(\chi_{\sigma_3}^s)^\vee}_{\in \mathfrak{p}^c}) \oplus (\mathrm{others}).$$

Assume $\bar{\rho}$ is semisimple and strongly generic.

Theorem 4 (BHHMS2)

As a G-representation, $\pi_v(\bar{r})$ can be generated by $D_0(\bar{\rho})$. $\cong \pi(\bar{r})^{K_1}$

The proof uses the computation of (φ, Γ)-modules attached to $\pi_v(\bar{r})$ (Lecture 4).

Remark. The non-semisimple case (under weaker genericity condition) will be treated in Lecture 9 (due to HW, the proof is of different nature).

Lemma 5

Let π' a subquotient of $(\pi_v(\bar{r}))^{\oplus r}$

(i) $\dim_{\mathbb{F}} \mathbb{V}(\pi') \leq m_{\mathfrak{p}_0}(\pi')$.

(ii) If π' is a subrepresentation of $\pi_v(\bar{r})$, then

$$\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi') = \underline{\lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi')}.$$

In particular, $\mathbb{V}(\pi') \neq 0$ if $\pi' \neq 0$.

(iii) If π' is a quotient of $\pi_v(\bar{r})$ and $\underline{\pi' \neq 0}$, then $\mathbb{V}(\pi') \neq 0$.

Subquot π'

$\overset{?}{\Rightarrow} V(\pi') = 0$

$\Rightarrow$ can't deduce. $\pi(\bar{r})$

finite length from

$\dim V(\pi(\bar{r})) < \infty$

Proof of Thm.4 Want to $\pi(\bar{r}) \supseteq \langle G. D_0(\bar{\rho}) \rangle = \pi_1$, π_2 = quotient

$\underline{\underline{\pi_2 = 0}}$

(ii) $\mathrm{Soc}\, \pi(\bar{r}) = \mathrm{Soc}(\pi_1)$

$\Rightarrow \dim V(\pi(\bar{r})) = \dim V(\pi_1)$

V exact $\Rightarrow V(\pi_2) = 0. \Rightarrow \pi_2 = 0.$ $\square$

Proof of Lemma 5. (i) $\underbrace{\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi')}_{\leq}$ if π' is subquot. of $\pi_v(\bar{r})$.

if $\pi' = \pi(\bar{r})$, have.

$$\dim V(\pi(\bar{r})) = 2^f = m_{\mathfrak{p}_0}(\pi(\bar{r}))$$

$\Rightarrow$ claim by dévissage. b/c $\mathbb{V}$ is exact.

(ii) $\dim_{\mathbb{F}} \mathbb{V}(\pi') = m_{\mathfrak{p}_0}(\pi') = \lg(\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_K)}\pi')$ for $\pi' \subset \pi_v(\bar{r})$.

$\Rightarrow \pi(\bar{r})^\vee \twoheadrightarrow \pi'^\vee$

① show $m_{\mathfrak{p}_0}(\pi') \leq \lg(\mathrm{soc}\,\pi')$.

$\mathcal{P}' = \{\chi = \sigma^{I_1} : \sigma \in \mathrm{soc}\,\pi'\}$

$$N = \Big(\bigoplus_{\chi\in\mathcal{P}'} N_\chi\Big) \oplus \Big(\bigoplus_{\chi\in\mathcal{P}'^c} N_\chi\Big) \twoheadrightarrow \mathrm{gr}(\pi(\bar{r})^\vee)$$

$$\Big(\bigoplus_{\chi\in\mathcal{P}'} N_\chi\Big) \oplus \underbrace{\Big(\cdots\Big)}_{} \twoheadrightarrow \mathrm{gr}(\pi'^\vee)$$

$m_{\mathfrak{p}_0}(-) = 0$

$\Rightarrow m_{\mathfrak{p}_0}(\pi') \leq m_{\mathfrak{p}_0}\big(\bigoplus_{\chi\in\mathcal{P}'} -\big) = |\mathcal{P}'| = \lg(\mathrm{soc}\,\pi')$

② Show $\dim \mathbb{V}(\pi') \geq \lg(\mathrm{soc}\,\pi')$.

$\tau_\ell \subseteq \mathrm{soc}(\pi')$

$\mathcal{S}$-cycle.

- $\tau_\ell = \bigoplus_{\substack{\sigma\in W(\bar{\rho}) \\ \mathrm{length}(\sigma)=\ell}} \sigma$, $\underline{\ell\in\{0,\ldots,f\}}$
- $\langle G.\tau_\ell\rangle \subseteq \pi'$

$\dim \mathbb{V}(\langle G.\tau_\ell\rangle) \geq \lg(\tau_\ell)$.

$\mathrm{soc}(\pi') \overset{?}{=} \bigoplus \tau_\ell$ ([BP])

$\Rightarrow$ (ii).

$M' \subseteq M$, ϵ j

$j(M') > j$

(iii) $\mathbb{V}(\pi') \neq 0$ for π' non-zero quotient of $\pi_v(\bar{r})$. $\to \pi' \to 0$

$0 \to \pi'' \to$

$\Rightarrow \quad 0 \to \pi'^{\vee} \to \pi(\bar{r})^{\vee} \to \pi''^{\vee} \to 0$

$E^i(-) \Rightarrow 0 \to E^{2f}(\pi''^{\vee}) \to \underbrace{E^{2f}(\pi(\bar{r})^{\vee})}_{CM} \xrightarrow{\gamma} E^{2f}(\pi'^{\vee}) \to E^{2f+1}(\pi''^{\vee}) \to 0$

Let $\tilde{\pi}' := (\mathrm{Im}(\gamma) \otimes \zeta^{-1})^{\vee}$

recall $\pi(\bar{r})^{\vee}$ self dual. $E^{2f}(\pi(\bar{r})^{\vee}) \otimes \zeta^{-1} \simeq \pi(\bar{r})^{\vee}$

$\Rightarrow \pi(\bar{r})^{\vee} \twoheadrightarrow \tilde{\pi}'^{\vee}$

$\Rightarrow \tilde{\pi}' \hookrightarrow \pi(\bar{r})$

Claim: $\tilde{\pi}' \neq 0 \quad (\Leftarrow \mathrm{Im}(\gamma) \neq 0)$

Pf: $\pi(\bar{r})^{\vee}$ is CM $\Rightarrow$ pure: any submod of $\pi(\bar{r})^{\vee}$ has the same grade

$\Rightarrow E^{2f}(\pi'^{\vee}) \neq 0$

$0 \to \mathrm{Im}(\gamma) \to \underbrace{E^{2f}(\pi'^{\vee})}_{\neq 0} \to E^{2f+1}(\pi''^{\vee}) \to 0$

• $j(E^{j(M)}(M)) = j \quad \Rightarrow j(\quad) = 2f$ • Auslander condition $\Rightarrow j(E^{2f+1}(-)) \geq 2f+1$.

Show: $m_{\mathfrak{p}}(\pi') = m_{\mathfrak{p}}(\tilde{\pi}')$ $\forall \mathfrak{p}$ of $gr(\Lambda)/J$.

in particular.

$$\dim V(\pi') \overset{(i)}{=} m_{\mathfrak{p}_0}(\pi') = m_{\mathfrak{p}_0}(\tilde{\pi}') \overset{(i)}{=} \dim V(\tilde{\pi}')$$

$\neq 0$

b/c $\tilde{\pi}' \hookrightarrow \pi(\bar{r})$ (ii).

Pf: Auslander condition:

$$\Rightarrow m_{\mathfrak{p}}(E^{2f}(\pi'^{\vee})) = m_{\mathfrak{p}}(\tilde{\pi}'^{\vee}) + m_{\mathfrak{p}}(\underline{E^{2f+1}})$$

fact:

$$m_{\mathfrak{p}}(E^{2f}(\pi'^{\vee})) = m_{\mathfrak{p}}(\pi'), \qquad m_{\mathfrak{p}}(E^{2f+1}) = 0.$$

$\dim \leq f-1$

$gr(\Lambda)/\mathfrak{p}$: dim f.

$\square$

We assume from now on $r = 1$ (NOT for simiplicity). even. $\pi(\bar{r})^{K_1} \cong D_0(\bar{\rho})^{\oplus r}$

input: $\pi(\bar{r}) = \langle G \cdot D_0(\bar{\rho}) \rangle$ $\nexists$ $\pi_v(\bar{r}) = (\ ?\ )^{\oplus r} \to \pi(\bar{\rho})$.

Theorem 5 (BHHMS2)

(i) If $\bar{\rho}$ is irreducible, then $\pi_v(\bar{r})$ is irreducible.

(ii) If $\bar{\rho}$ is reducible split, then

$$\pi_v(\bar{r}) \cong \pi_0 \oplus \pi' \oplus \pi_f$$

$\pi' = \bigoplus_{\ell=1}^{f-1} \pi_\ell \leftarrow$ s.s.

with π_0, π_f principal series. If moreover $f = 2$, then π' is (irreducible) supersingular.

Remark

(i) In general, it is not clear if "finite generated $\Longrightarrow$ finite length".

(ii) [BP, Thm. 19.10(ii)] (which says that if $\bar{\rho}$ is reducible split then π in their construction is also semisimple) does not apply here (cf. Lecture 3).

Proof of Thm.5 (i) Follows from Lecture 3, once we know $\pi_v(\bar{r})$ is generated by $D_0(\bar{\rho})$ by **Theorem 4**.

[BP]. any such rep. generated by $D_0(\bar{\rho})$ is irred + s.s.

(ii) $\bar{\rho}$ is red split:

$\sigma_0 = (r_0, \ldots, r_{f-1})$, $\sigma_f = (p-3-r_0, \ldots, p-3-r_{f-1})$

Let $\pi_0 := \langle G \cdot \sigma_0 \rangle$.

Claim. π_0 is PS. (using weight cycling + $JH(\mathrm{Ind}_I^{GL_2(O_K)} \chi_{\sigma_0}) \cap W(\bar{\rho}) = \{\sigma_0\}$)

Sim. π_f is PS

$\underbrace{\pi_0 \oplus \pi_f}_{\text{Self-duality.}} \hookrightarrow \pi(\bar{r}) \twoheadrightarrow \pi_0 \oplus \pi_f.$

an isom.

$\pi_0 \oplus \pi_f \longrightarrow ? \longrightarrow \pi_0 \oplus \pi_f$

Need.

Use: $\pi(\bar{r}) = \langle G \cdot D_0(\bar{\rho}) \rangle = \langle G \cdot \underbrace{\mathrm{soc}\, \pi(\bar{r})}_{= \bigoplus_{\sigma \in W(\bar{\rho})} \sigma} \rangle$

$$\iota : \bigoplus \sigma \hookrightarrow \pi(\bar{r}) \twoheadrightarrow \pi_0 \oplus \pi_f \twoheadrightarrow \pi_0$$

non-zero!

$\bigoplus_{\sigma' \neq \sigma_0} \sigma' \longrightarrow 0$ inside π_0

$\mathrm{Im}(\iota) \subseteq \mathrm{soc}(\pi_0) = \sigma_0$

$$\iota|_{\sigma_0} : \sigma_0 \dashrightarrow \pi_0 \oplus \pi_f$$

$$\subseteq \mathrm{soc}(-) = \sigma_0 \oplus \pi_f$$

$\Rightarrow$ ι induces an isom $\sigma_0 \xrightarrow{\sim} \sigma_0 = \mathrm{soc}(\pi_0)$.

$\Rightarrow$ BL.: ι induces an isom $\pi_0 \xrightarrow{\sim} \pi_0$,

$\pi_f \xrightarrow{\sim} \pi_f$.

$\Rightarrow \pi(\bar{r}) = (\pi_0 \oplus \pi_f) \oplus \pi'$.

If $f=2$, BP $\Rightarrow \mathrm{soc}(\pi') = (\sigma_1 \oplus \sigma_1^{[s]}) \Rightarrow \pi'$ is irred and s.s. $\square$

Some facts for later use :

$\mathbb{M}_\infty$ defines an exact functor : $\mathcal{O}[\mathrm{GL}_2(\mathcal{O}_K)]$-**Mod** $\longrightarrow R_\infty$-**Mod**

$$M_\infty(\Theta) := \mathrm{Hom}^{\mathrm{cont}}_{\mathrm{GL}_2(\mathcal{O}_K)}(\mathbb{M}_\infty, \Theta^d)^d.$$

(a) For $\lambda = (a_j, b_j)_{0\leq j\leq f-1}$ with $a_j > b_j$, and $\tau : I_K \to \mathrm{GL}_2(E)$ an inertial type, set

$$V(\lambda-\eta) := \bigotimes_{0\leq j\leq f-1} \left((\mathrm{Sym}^{a_j-b_j-1}E^2)\otimes \det{}^{b_j}\right)^{\mathrm{Fr}^j}$$

with $\eta = (1,0)$, and $\sigma(\tau) :=$ smooth irred. rep (over E) of $\mathrm{GL}_2(\mathcal{O}_K)$ by Henniart's inertial LLC.

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If $\Theta \subset V(\lambda-\eta)\otimes\sigma(\tau)$ is an $\mathcal{O}[\mathrm{GL}_2(\mathcal{O}_K)]$-lattice, then $M_\infty(\Theta)$ is maximal CM and the action of R_∞ factors through $R_\infty \otimes_{R_{\bar{\rho}}^{\square}} R_{\bar{\rho}}^{\lambda,\tau}$, where $R_{\bar{\rho}}^{\lambda,\tau}$ is Kisin's pot. semistable deformation ring of type (λ,τ).

(b) $M_\infty(\Theta)/\mathfrak{m}_\infty \cong \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta, \pi_v(\bar{r}))^\vee$.

In particular, $M_\infty(\Theta)$ is a cyclic R_∞-module iff

$$\dim_{\mathbb{F}} \mathrm{Hom}_{\mathrm{GL}_2(\mathcal{O}_K)}(\Theta/\varpi\Theta, \pi_v(\bar{r})) = 1.$$

Example. $[\pi_v(\bar{r})^{K_1} : \sigma] = 1$ if and only if $M_\infty(\mathrm{Proj}_{\mathrm{GL}_2(\mathbb{F}_q)}\sigma)$ is cyclic R_∞-module.

(c) The flatness of M_∞ over R_∞ induces a Koszul type resolution of $\pi_v(\bar{r})^\vee$ in terms of M_∞ :

$$\cdots \to M_\infty^{\oplus\binom{n}{2}} \to M_\infty^{\oplus n} \to M_\infty \to \pi_v(\bar{r})^\vee \to 0.$$

Thank you !