Multivariable $(\varphi, \mathcal{O}_K^\times)$-modules

p prime number K unr $/\mathbb{Q}_p$ degree f

$\mathbb{F}_q$ residue f. of K $\mathbb{F}/\mathbb{F}_q$ finite

Convention: increasing filtrations M filtered ab group

$(F_n M)_{n\in\mathbb{Z}}$ $F_n M \subset F_{n+1} M$

$gr(M) = \bigoplus_{n\in\mathbb{Z}} gr_n(M)$ $gr_n(M) = F_n M / F_{n-1} M$

1. Completed group algebra of the pro-p Iwahori

$I_1 = \begin{pmatrix} 1+p\mathcal{O}_K & \mathcal{O}_K \\ p\mathcal{O}_K & 1+p\mathcal{O}_K \end{pmatrix} \subset GL_2(K)$ $Z_1 = (1+p\mathcal{O}_K)\, Id.$

$G = I_1 / Z_1$ $\underset{\underset{m_G}{\cup}}{\mathbb{F}[\![G]\!]}$ local complete noetherian algebra.

m_G maximal ideal.

Rk π smooth rep of $GL_2(K)$ with central character

$$\begin{tikzcd} I_1 \arrow[dr] \arrow[r, phantom, "\curvearrowright"] & \pi \\ & G \curvearrowright \end{tikzcd}$$

$\mathbb{F}[\![G]\!]$ filtered $F_n(\mathbb{F}[\![G]\!]) = \begin{cases} m_G^{-n} & \text{if } n \le 0 \\ \mathbb{F}[\![G]\!] & \text{if } n \ge 0. \end{cases}$

Theorem (Clozel, Lazard) Assume $p > 2$.

$gr\,\mathbb{F}[\![G]\!] \simeq \bigotimes_{j=0}^{f-1} U(g_j)$ $g_j = \mathbb{F} y_j \oplus \mathbb{F} z_j \oplus \mathbb{F} h_j.$

$[y_j, z_j] = h_j$ h_j central.

$\deg(y_j) = \deg(z_j) = -1 \rightarrow \deg h_j = -2$

[Lazard: H pro-p-group + p-saturated valuation

$gr\,\mathbb{F}[\![G]\!] \simeq U(\underset{\uparrow}{\text{Lie}\, H} \otimes_{\mathbb{Z}_p} \mathbb{F})$

$\mathbb{Z}_p$-Lie algebra of H.

$\mathbb{F} \otimes_{\mathbb{Z}_p} \text{Lie}\, G \simeq \bigoplus_{j=0}^{f-1} g_j$

$\mathbb{F}_q \xhookrightarrow{\sigma_j} \mathbb{F}$

$y_j \longleftrightarrow \sigma_j \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $h_j \longleftrightarrow \sigma_j \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}$

$z_j \longleftrightarrow \sigma_j \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$]

$J \subset \mathrm{gr}\,\mathbb{F}[\![G]\!]$ left ideal generated by $h_j,\ y_j z_j,\ 0 \leq j \leq f-1$
2-sided ideal

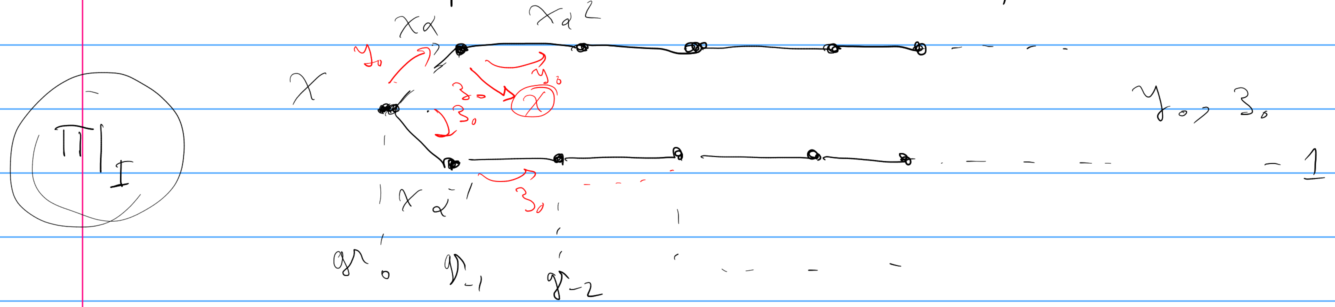
$$\mathrm{gr}\,\mathbb{F}[\![G]\!] \big/ J \simeq \underbrace{\mathbb{F}[y_0, z_0, y_1, z_1, \dots] \big/ (y_j z_j)}_{f\text{- dim complete intersection}}$$

$\pi \in \mathrm{Rep}^{\mathrm{adm}}\, GL_2(K)$ with central char.
$\hookrightarrow \pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F})$ fg $\mathbb{F}[\![G]\!]$-module
filtered for $\mathfrak{m}_G$-adic filtration.

$\mathrm{gr}(\pi^\vee) = \bigoplus_{n \leq 0} \mathrm{gr}_n(\pi^\vee)$ $\mathrm{gr}_n(\pi^\vee) = \mathfrak{m}_G^{-n}\pi^\vee \big/ \mathfrak{m}_G^{-n+1}\pi^\vee$
$\hookrightarrow$ fg $\mathrm{gr}(\mathbb{F}[\![G]\!])$-module.

Def: $\mathcal{C}_1$ = category of sm adm rep of $GL_2(K)$, + central char st $\exists\, m \geq 0$ st $J^m\, \mathrm{gr}(\pi^\vee) = 0$.

Remarks 1) $\mathcal{C}_1$ abelian category.
2) $K = \mathbb{Q}_p$. $\mathcal{C}_1 \supset$ irreducible representation.



3) $\pi \in \mathcal{C}_1 \implies GK\dim(\pi) \leq f = [K:\mathbb{Q}_p]$.

$\Big[$ Krull dim $\left(\mathrm{gr}\,\mathbb{F}[\![G]\!] \big/ J\right) = f \implies j_{\mathrm{gr}(\mathbb{F}[\![G]\!])}(\mathrm{gr}(\pi^\vee)) \geq 2f = \dim G - f$.

$\Downarrow$ Björk

$j_{\mathbb{F}[\![G]\!]}(\pi^\vee) \geq 2f$

$\Downarrow$

$GK\dim(\pi^\vee) \leq f$ $\Big]$

Thm With notation and assumptions of lecture 1.

$\underset{\substack{\circlearrowleft \\ GL_2(K)}}{\Pi_v(\bar{r})} \in \mathcal{C}_1$.

$\implies \mathcal{C}_1$ contains enough objects of global origin.

2. $\mathcal{C}_1 \longrightarrow (\varphi, \mathcal{O}_K^\times)$-modules

$$\mathbb{F}[[N_0]] \simeq \mathbb{F}[[\mathcal{O}_K]] \simeq \mathbb{F}[[Y_0, \dots, Y_{f-1}]]$$

$N_0 = \begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix} \simeq \mathcal{O}_K$ $\qquad$ $Y_i = \sum_{a \in \mathbb{F}_q^\times} a^{-p^i} \begin{pmatrix} 1 & [a] \\ & 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} [\mathbb{F}_q^\times] & 0 \\ 0 & [\mathbb{F}_q^\times] \end{pmatrix}$.

$\cup$

$m_{N_0} \ni (Y_0, \dots, Y_{f-1})$.

$y_i = Y_i \bmod m_G^2$. $\qquad$ $\mathrm{gr}\, \mathbb{F}[[N_0]] \simeq \mathbb{F}[y_0, \dots, y_{f-1}]$.

$v : \mathbb{F}[[N_0]] \longrightarrow \mathbb{Z} \cup \{+\infty\}$ $\quad$ m_{N_0}-adic valuation

$v(x) = \max \{ i \mid x \in m_{N_0}^i \}$.

$F_n \mathbb{F}[[N_0]] = v^{-1}([-n, +\infty])$.

→ $v(ab) = v(a) + v(b)$

$S = \{ (Y_0 \cdots Y_{f-1})^n \mid n \geq 0 \}$ $\qquad$ $\mathbb{F}[[N_0]]_S \xrightarrow{v_S} \mathbb{Z} \cup \{+\infty\}$

$A = \widehat{R_S}$ ⟵ completion w.r.t. v_S.

$v_A : A \longrightarrow \mathbb{Z} \cup \{+\infty\}$.

$F_n A = v_A^{-1}([-n, +\infty])$.

$\mathrm{gr}(A) \simeq \mathrm{gr}(\mathbb{F}[[N_0]]_S) \simeq \mathrm{gr}(\mathbb{F}[[N_0]])[(y_0 \cdots y_{f-1})^{-1}]$.

<u>Additional structures?</u> $\begin{pmatrix} p & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}$ on N_0.

$\varphi : \mathbb{F}[[N_0]] \hookrightarrow \mathbb{F}[[N_0]]$ $\quad$ finite flat of degree p^f.

$R = \mathbb{F}[[N_0]]$ $\}$ $\varphi(Y_i) = Y_{i-1}^p$ $\qquad$ $\varphi : \mathbb{F}[[N_0]]_S \hookrightarrow \mathbb{F}[[N_0]]_S \rightsquigarrow \varphi : A \hookrightarrow A$

finite flat of degree p^f.

$\mathcal{O}_K^\times \ni a$ $\qquad$ $\begin{pmatrix} a & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} a & \\ & 1 \end{pmatrix}^{-1}$ on N_0.

$\gamma_a : \mathbb{F}[[N_0]] \xrightarrow{\sim} \mathbb{F}[[N_0]]$. $\qquad$ γ_a extends to $A \overset{\gamma_a}{\xrightarrow{\sim}} A$.

$\gamma_a(Y_0 \cdots Y_{f-1}) \in A^\times$

$\notin \mathbb{F}[[N_0]]_S$.

$\mathcal{O}_K^\times \longrightarrow \mathrm{Aut}(A)$

φ and $\mathcal{O}_K^\times$ commute.

<u>Def</u>: A $(\varphi, \mathcal{O}_K^\times)$-module (over A) is a f.g. A-module D

$\varphi : D \longrightarrow D$ $\quad$ φ-semilinear $\Big)$ commute

$\mathcal{O}_K^\times \longrightarrow \mathrm{Aut}(D)$ $\quad$ $\mathcal{O}_K^\times$-semilinear $\Big)$

$A \otimes_{A,\varphi} D \xrightarrow{1\otimes\varphi} D$ A-linear

D is étale if $1\otimes\varphi$ is an iso.

Proposition: M is a fg A-module + $\mathcal{O}_K^\times$-semilinear action of $\mathcal{O}_K^\times$ then M is projective as A-module

lemma: The $\mathcal{O}_K^\times$-stable ideals of A are A and 0.

[$\mathfrak{a} \subset A$ $\mathcal{O}_K^\times$-stable ideal $\mathfrak{a} = A\cdot(\mathfrak{a}\cap\varphi(A))$.

$\mathfrak{a} = \bigcap_{n\geq 0} A\cdot(\mathfrak{a}\cap\varphi^n(A)). = 0$ if $\mathfrak{a} \neq A$.]

Prop: M fg A-module + $\mathcal{O}_K^\times$ $\curvearrowright$.

$$E_2^{p,q} = \underbrace{\mathrm{Ext}_A^p(\mathrm{Ext}_A^{-q}(M,A))}_{\text{torsion } A\text{-modules} + fg + \mathcal{O}_K^\times \text{ if } p \text{ or } q>0.} \Longrightarrow M.$$

$\Longrightarrow \mathrm{Ann}(E_2^{p,q}) \subset A$ $\circlearrowleft_{\mathcal{O}_K^\times}$ $\overset{\shortparallel}{A}$ if $p,q>0$

$\rightarrow M = M^{\vee\vee}$ $\mathrm{Ext}_A^p(M^\vee, A) = 0 \rightarrow M^\vee$ proj. $\rightarrow M$ proj. .

Specialization: $\mathrm{Tr} : N_0 \twoheadrightarrow \mathbb{Z}_p$

$$\begin{tikzcd}
N_0 \arrow[r, two heads] \arrow[d, phantom, "\wr"] & \mathbb{Z}_p & \mathbb{F}[\![N_0]\!] \arrow[r, "\mathrm{Tr}"] & \mathbb{F}[\![\mathbb{Z}_p]\!] \arrow[d, phantom, "\wr"] \\
\mathcal{O}_K & & & \mathbb{F}[\![X]\!] \arrow[d, phantom, "\cap"] \\
 & & A \arrow[uu, phantom, "\cap"] \arrow[r, "\mathrm{Tr}"] & \mathbb{F}(\!(X)\!)
\end{tikzcd}$$

φ-equivariant

$\Gamma = \mathbb{Z}_p^\times \subset \mathcal{O}_K^\times$

$\varphi(X) = X^p$

D $(\varphi, \mathcal{O}_K^\times)$-module $\rightsquigarrow D \otimes_{A,\mathrm{Tr}} \mathbb{F}(\!(X)\!)$ (φ, Γ)-module

étale if D is étale

Theorem: ∃ exact functor $\mathcal{C}_1^{or} \xrightarrow[D_A^{et}]{} \left(\begin{matrix}\text{étale}\\ (\varphi, \mathcal{O}_K^\times)\text{-modules}\end{matrix}\right)$ such that

$$D_A^{et} \otimes_{A,\mathrm{Tr}} \mathbb{F}(\!(X)\!) \simeq D_\xi^\vee\big|_{\mathcal{C}_1}.$$

Corollary: $\pi \in \mathcal{C}_1$, $\dim_{\mathbb{F}(\!(X)\!)} D_\xi^\vee(\pi) < +\infty$.

Construction: $\pi \in \mathcal{C}_1$. $\pi^\vee$ filtered $\mathbb{F}[\![\overset{I_1}{\overset{\downarrow}{G}}]\!]$

↳ $\mathfrak{m}_G$-adic topology.

$N_0 \hookrightarrow G \rightarrow \mathbb{F}[\![N_0]\!] \rightarrow \mathbb{F}[\![G]\!]$

$\pi^\vee$ filtered $\mathbb{F}[\![N_0]\!]$-module. (not $\mathfrak{m}_{N_0}$-adic topology!).

$S \subset \mathbb{F}[\![N_0]\!]$ $\qquad \pi_S^\vee = \mathbb{F}[\![N_0]\!]_S \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee$

filtered for the "tensor product filtration".

$$F_n(\pi_S^\vee) = \varinjlim_{s \gg 0} (Y_0 Y_1 \cdots Y_{f-1})^{-s} \, \mathfrak{m}_{\underline{G}}^{sf-n} \, \pi^\vee$$

$$D_A(\pi) = \widehat{\pi_S^\vee} = \varprojlim_n \pi_S^\vee / F_n(\pi_S^\vee) \quad \Big(\simeq \widehat{A \otimes_{\mathbb{F}[\![N_0]\!]} \pi^\vee} \Big).$$

$\circlearrowleft$

$\mathcal{O}_K^\times$ semilinear.

Pb : $\varphi \longleftrightarrow \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \pi_S^\vee \to \pi_S^\vee$

Define a ψ: $\qquad \pi \xrightarrow[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]{\sim} \pi \qquad \varphi : \pi^\vee \xrightarrow{\sim} \pi^\vee$

$\qquad \lambda \longmapsto \lambda \circ \begin{pmatrix} p & \\ & 1 \end{pmatrix}$

$\boxed{\psi(\varphi(a) v) = a \psi(v)}$ $\qquad v \in \pi^\vee \quad a \in \mathbb{F}[\![N_0]\!]$.

ψ $\qquad \pi_S^\vee \longrightarrow \pi_S^\vee$

$$\frac{x}{(Y_0 \cdots Y_{f-1})^{pn}} \longmapsto \frac{\psi(x)}{(Y_0 \cdots Y_{f-1})^n}$$

Fact ψ continuous.

$\rightsquigarrow \quad \psi : D_A(\pi) \longrightarrow D_A(\pi)$

linearize it

$$\beta : D_A(\pi) \longrightarrow A \otimes_{A,\varphi} D_A(\pi)$$

$$v \longmapsto \sum_{g \in N_0/N_0^p} \delta_g \otimes \psi(\delta_g^{-1} v).$$

commutes to $\mathcal{O}_K^\times$.

A-linear.

If β is an iso, we can define $\varphi : D_A(\pi) \to D_A(\pi)$

$1 \otimes \varphi = \beta^{-1}$.

$$D_A(\pi) \xrightarrow{\beta} A \otimes_{A,\varphi} D_A(\pi) \xrightarrow{1 \otimes \beta} A \otimes_{A,\varphi^2} D_A(\pi) \longrightarrow - - - - -$$

$$D_A^{et}(\pi) = \varinjlim_{1 \otimes_{\varphi^n} \beta} A \otimes_{A,\varphi^n} D_A(\pi)$$

$$\begin{tikzcd}
\underline{\beta} : D_A^{et}(\pi) \arrow[r, "\sim"] \arrow[d, phantom, "\|"] & A \otimes_\varphi D_A^{et}(\pi) \arrow[d, phantom, "\wr"] \\
\varinjlim A \otimes_{\varphi^n} D_A(\pi) \arrow[r, "1 \otimes_{\varphi^n} \beta"'] & \varinjlim_n A \otimes_{\varphi^{n+1}} D_A(\pi)
\end{tikzcd}$$

$D_A^{et}(\pi) \circlearrowleft \varphi$ st $1 \otimes \varphi = \underline{\beta}^{-1}$.

Proposition $D_A(\pi)$ is a fg A-module, $D_A^{et}(\pi)$ is a quotient of $D_A(\pi)$.

Proof: $\pi \in \mathcal{C}_1$. Assume $J \, \mathrm{gr}(\pi^\vee) = 0$.

$D_A(\pi)$ is a filtered A-module

$$\operatorname{gr} D_A(\pi) \simeq \operatorname{gr}(\pi_S^\vee) \simeq \operatorname{gr}(\pi^\vee)\underbrace{[(y_0\cdots y_{f-1})^{-1}]} \simeq \operatorname{gr}(A) \otimes_{\operatorname{gr}\mathbb{F}[\![N_0]\!]} \operatorname{gr}(\pi^\vee) \nearrow$$

$$\simeq \underbrace{\left(\operatorname{gr}(A) \otimes_{\operatorname{gr}\mathbb{F}[\![N_0]\!]} \operatorname{gr}\mathbb{F}[\![G]\!]/J\right)} \otimes_{\underbrace{\operatorname{gr}\mathbb{F}[\![G]\!]}} \operatorname{gr}(\pi^\vee)$$

$$\operatorname{gr}\mathbb{F}[\![G]\!]/J \simeq \mathbb{F}[y_0, z_0, y_1, z_1, \dots]/(y_i z_i)$$

$\uparrow$ $\quad * \in \mathbb{F}^\times y_i$

$$\operatorname{gr}\mathbb{F}[\![N_0]\!] = \mathbb{F}[y_0, \dots, y_{f-1}] \quad y_i$$

$N_0 \hookrightarrow G$

$\mathbb{F}[\![N_0]\!] \hookrightarrow \mathbb{F}[\![G]\!]$

$$\rightsquigarrow \operatorname{gr} A \otimes_{\operatorname{gr}\mathbb{F}[\![N_0]\!]} \operatorname{gr}\mathbb{F}[\![G]\!]/J \simeq \mathbb{F}[y_0 \dots]/(y_i z_i)\,[y_0^{-1}, \dots, y_{f-1}^{-1}]$$

$$\simeq \mathbb{F}[y_0, \dots, y_{f-1}, (y_0\cdots y_{f-1})^{-1}] \simeq \operatorname{gr}(A).$$

$\operatorname{gr}(\pi^\vee)$ f g $\operatorname{gr}\mathbb{F}[\![G]\!]$ $\rightsquigarrow$ $\operatorname{gr} D_A(\pi)$ f g over $\operatorname{gr}(A)$

$\rightsquigarrow$ $D_A(\pi)$ f g A-module.

$$\begin{tikzcd}
D_A(\pi) \arrow[r, "\beta_m"] \arrow[dr, "\beta_n"'] & A \otimes_{A,\varphi^m} D_A(\pi) \arrow[d] \\
& A \otimes_{A,\varphi^n} D_A(\pi)
\end{tikzcd}$$

$m \leq n.$

$(\operatorname{Ker}\beta_m) \nearrow$

$N = \bigcup \operatorname{Ker}\beta_m$ sub A-module of $D_A(\pi)$.

$$\beta(N) \subset A \otimes_{A,\varphi} N \subset A \otimes_{A,\varphi} D_A(\pi).$$

$$\beta : D_A(\pi)/N \overset{\sim}{\hookrightarrow} A \otimes_{A,\varphi} D_A(\pi)/N. \quad \circlearrowleft \mathcal{O}_K^\times$$

$\rightarrow$ $\operatorname{Coker}\beta$ is a torsion f g A-module $+ \mathcal{O}_K^\times$-action.

$\operatorname{Coker}\beta = 0.$

$$D_A^{et}(\pi) \simeq \varinjlim_m A \otimes_{A,\varphi^m} D_A(\pi) \xrightarrow{\sim} \varinjlim_m A \otimes_{A,\varphi^m} D_A(\pi)/N \simeq D_A(\pi/N). \quad \square.$$

<u>**Remark**</u>: $\operatorname{rk}_A D_A^{et}(\pi) \leq \operatorname{rk}_A D_A(\pi) = \underbrace{\operatorname{rk}_{\operatorname{gr} A} \operatorname{gr} D_A(\pi)}$

$$\operatorname{gr} D_A(\pi) \simeq \operatorname{gr}(\pi^\vee)[(y_0\cdots y_{f-1})^{-1}].$$

Assume that $J \operatorname{gr}(\pi^\vee) = 0$.

$$\operatorname{Spec}\left(\operatorname{gr}\mathbb{F}[\![G]\!]/J\right) = \bigcup_{2^f} V(y_i, z_j, i \in T, j \notin T) \quad T \subset \{0, \dots, f-1\}.$$

$\rightsquigarrow \quad \mathfrak{p}_0 = (\mathfrak{z}_0, \cdots, \mathfrak{z}_{f-1}).$

$$\mathrm{rk}_{\mathrm{gr} A}\, \mathrm{gr}\, D_A(\pi) = \dim_{k(\mathfrak{p})}\, \mathrm{gr}(\pi^\vee) \otimes k(\mathfrak{p}_0)$$

When $J^m \mathrm{gr}(\pi^\vee) = 0.$ $\quad m_{\mathfrak{p}_0}(\pi^\vee) = \sum_{k \geq 0} \dim_{k(\mathfrak{p}_0)} \left(\frac{\mathfrak{p}_0^k\, \mathrm{gr}(\pi^\vee)}{\mathfrak{p}_0^{k+1}\, \mathrm{gr}(\pi^\vee)} \right) \otimes k(\mathfrak{p}_0)$

$$\rightsquigarrow \quad \mathrm{rk}_A\, D_A^{et}(\pi) \leq m_{\mathfrak{p}_0}(\mathrm{gr}(\pi^\vee)).$$

$$D_A^{et} \otimes_{T\Lambda} \mathbb{F}((X)) \underset{\twoheadrightarrow}{\simeq} D_\xi^\vee \quad \rightsquigarrow \quad \boxed{\begin{gathered} \dim_{\mathbb{F}((X))} D_\xi^\vee(\pi) \leq m_{\mathfrak{p}_0}(\pi^\vee). \\ \pi \in \mathcal{C}_1. \end{gathered}}$$

$D_A(\pi) \qquad D_A^{et}(\pi) \qquad \Psi.\ \Pi^\vee \xrightarrow{\sim} \pi^\vee$