Shimura curves, Gelfand-Kirillov dimension

1. Shimura curves

F tot real $\qquad D$ quaternion center F

$\exists!\, \tau : F \hookrightarrow \mathbb{R}$ st $D \otimes_{F,\tau} \mathbb{R} \simeq M_2(\mathbb{R})$

$G_{/F} \qquad G(R) \simeq (D \otimes_F R)^\times$

$G(F \otimes_{\mathbb{Q}} \mathbb{R}) \simeq GL_2(\mathbb{R}) \times (\mathbb{H}^\times)^{d-1} \qquad d = [F:\mathbb{Q}]$

$\downarrow$

$GL_2(\mathbb{R}) \curvearrowright \mathbb{C} \setminus \mathbb{R}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \dfrac{az+b}{cz+d}$

$U \subset G(\mathbb{A}_F^\infty) \qquad \mathbb{A}_F^\infty = \prod'_{v \text{ finite places}} F_v$

compact open subgroup

$G(F) \backslash \left((\mathbb{C} \setminus \mathbb{R}) \times G(\mathbb{A}_F^\infty)/U \right)$ compact Riemann surface.

Shimura $\rightsquigarrow \exists\, X_{U/F}$ proper smooth curve$_{/F}$, st $X_U(\mathbb{C}) \simeq$
$F \hookrightarrow \mathbb{C}$

$g \in G(\mathbb{A}_F^\infty) \qquad U, V$ levels $\quad g^{-1} V g \subset U.$

$\exists!\; X_V \xrightarrow[/F]{} X_U$ given $\cdot g$ on $\mathbb{C}$-points.

$G(\mathbb{A}_F^\infty)$ "action" on tower of Shimura curves.

p prime $\mathbb{F}/_{\mathbb{F}_p}$ finite

$\boxed{\widetilde{H}^1(\mathbb{F})} = \varinjlim_U H^1_{ét}(X_U \times_F \bar{F}, \mathbb{F})$

$X_V \to X_U \qquad V \subset U$

$G(\mathbb{A}_F^\infty) \times Gal(\bar{F}/F)$

smooth representation: all stabilizers are open subgroup.

$[\bar{r} : Gal(\bar{F}/F) \to GL_2(\mathbb{F})$ abs irred + totally odd

$(\det(\bar{r}(c)) = -1 \quad \forall c$ complex conj$).$

$\pi(\bar{r}) = \mathrm{Hom}^{cts}_{Gal(\bar{F}/F)}(\bar{r}, \widetilde{H}^1(\mathbb{F})) \circlearrowleft G(\mathbb{A}_F^\infty)$ smooth.

$\wr\wr$

$\prod'_{v \text{ finite places of } F} G(F_v)$

$\bar{\rho}_v = \bar{\rho}|_{Gal(\bar{F}_v/F_v)}$

<u>Expectation</u>: (modularity + local-global compatibility).

$\Pi(\bar{\rho}) \neq 0$

$\forall v$, $\exists$ $\pi(\bar{\rho}_v)$ smooth representation of $G(F_v)$

(i) $\underline{\pi(\bar{\rho}) \simeq \bigotimes_v{}' \pi(\bar{\rho}_v)}$

(ii) $\bar{\rho}, \bar{\rho}'$ rep of $Gal(\bar{F}_p)$ as above

$$\pi(\bar{\rho}_v) \underset{G(F_v)}{\simeq} \pi(\bar{\rho}'_v) \iff \bar{\rho}_v \simeq \bar{\rho}'_v$$

$\bar{\rho}_v \longmapsto \pi(\bar{\rho}_v)$ mod p local Langlands correspondence.

<u>Remarks</u>: 1) $v \nmid p$, $\pi(\bar{\rho}_v)$ predicted (Buzzard-Diamond-Jarvis, Vigneras, Emerton-Helm)

2) $v | p$, $\pi(\bar{\rho}_v)$? F_v unr D split at v

$G(F_v) \simeq GL_2(F_v)$ (Breuil - Paškūnas)

3) $F_v = \mathbb{Q}_p$, D split at v $GL_2(\mathbb{Q}_p)$.

$\bar{\rho}_v \longmapsto \pi(\bar{\rho}_v)$ exists (Breuil, Colmez, Paškūnas)

4) $D \hookrightarrow M_2(\mathbb{Q}) \rightsquigarrow$ modular curves.

expectation proven (Emerton)

$v | p$ F unramified at v + D split at v.

$$G(F_v) \simeq GL_2(F_v) \simeq GL_2(\mathbb{Q}_{p^f}) \qquad f = [F_v : \mathbb{Q}_p].$$

$U^v \subset G(\mathbb{A}_F^{\infty, v})$ compact open subgroup. "tame level"

$$\Pi_v(\bar{\rho}) := \Pi(\bar{\rho})^{U^v} \circlearrowleft G(F_v) \simeq GL_2(F_v) \text{ smooth.}$$

Expectation $\Rightarrow$ $\underline{\pi_v(\bar{\rho})} \simeq \pi(\bar{\rho}_v)^{\oplus s}$ $s \geq 1$ U^v small enough.

$$s = \dim_{\mathbb{F}} \left(\bigotimes_{w \neq v}{}' \pi(\bar{\rho}_w) \right)^{U^v}.$$

<u>Results</u>: V_{GL_2} : rep $GL_2(F_v) \longmapsto$ rep of $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ / $\mathbb{F}$

lecture 4.

- Under "some" Hyp $V_{GL_2}(\pi_v(\bar{\rho})) \simeq \left(\mathrm{Ind}_{F_v}^{\otimes \mathbb{Q}_p}(\bar{\rho}_v)\right)^{\oplus s}$ // lect 4, 5, 6.
- $s = 1$. $\bar{\rho}_v$ is abs irred $\Rightarrow$ $\Pi_v(\bar{\rho})$ is irreducible.) lect 6
- $s = 1$, $f \leq 2$, $\Pi_v(\bar{\rho})$ finite length.
- computation of Gelfand-Kirillov dimension)) $\longleftarrow$ lect 7, 8, 9.

$GL_2(F_v)$

2. Gelfand-Kirillov dimension (all rep are over $\mathbb{F}$)

H p-adic Lie group. π smooth rep of H.

π admissible iff $\forall H' \subset H$ open subgroup,

$$\dim_{\mathbb{F}} \pi^{H'} < \infty.$$

<u>Remarks</u>: $\Pi_v(\bar{\rho})$ is admissible.

$U_v \subset GL_2(F_v)$ $\quad \Pi_v(\bar{\rho})^{U_v} \hookrightarrow H^1_{et}(X_{U^v \times U_v} \times \bar{F}, \mathbb{F})$.

(if U^v small enough)

Assume that H is a pro-p-group. (+ p-adic ~~Lie group~~)

$$\underset{\underset{m_H}{\cup}}{\mathbb{F}[\![H]\!]} = \varprojlim_{\substack{H' \triangleleft H \\ \text{open}}} \mathbb{F}[H/H'] \quad \text{local complete algebra}$$

$\hookrightarrow m_H$-adic topology.

Lazard $\rightsquigarrow$ $\mathbb{F}[\![H]\!]$ noetherian.

π smooth rep of H. $\quad \pi \quad \mathbb{F}[\![H]\!]$-module

$\pi^\vee = \mathrm{Hom}_{\mathbb{F}}(\pi, \mathbb{F}) \quad \mathbb{F}[\![H]\!]$-module.

Vigneras: π admissible $\iff$ $\pi^\vee$ f.g. $\mathbb{F}[\![H]\!]$-module.

(use Nakayama).

<u>Example</u>: H commutative analytic pro-p-group, torsion free

$H \simeq \mathbb{Z}_p^r \quad \mathbb{F}[\![H]\!] \simeq \mathbb{F}[\![X_1, \dots, X_r]\!] \rightsquigarrow$ regular.

M f.g. ~~module over $\mathbb{F}[\![H]\!]$~~

$$\operatorname{codim}_{\operatorname{Spec}\mathbb{F}[\![H]\!]} \operatorname{Supp}(M) = \min\{ i \mid \operatorname{Ext}^i(M, \mathbb{F}[\![H]\!]) \neq 0 \}$$

<u>Def</u>: H analytic pro-p-group, M f.g. $\mathbb{F}[\![H]\!]$-mod

$$j_H(M) = \min\{ i \mid \operatorname{Ext}^i(M, \mathbb{F}[\![H]\!]) \neq 0 \}.$$

<u>Properties</u>: 1) $H' \subset H$ open, $j_{H'}(M|_{\mathbb{F}[\![H']\!]}) = j_H(M)$

Venjakob
2) $M \neq 0$, $\quad 0 \leq j_H(M) \leq \dim H$.
3) $0 \to M' \to M \to M'' \to 0$ $\quad j_H(M) = \min(j_H(M'), j_H(M''))$.

<u>Def</u>. H analytic pro-p-group, $\pi \in \mathrm{Rep}^{adm}_{\mathbb{F}} H$, $\mathrm{GKdim}(\pi) = \dim(H) - j_H(\pi^\vee)$

H p-adic analytic group, $\quad$ ——— $\mathrm{GKdim}(\pi) = \mathrm{GKdim}(\pi|_{H_0})$

$H_0 \subset H$ open pro-p-subgroup (exist ← Lazard)

Remarks: 1) Lazard $\exists\, H_0 \subset H$ open pro-p-subgroup.

st $\mathrm{gr}(\mathbb{F}[\![H_0]\!]) = \bigoplus_{n\geq 0} \mathfrak{m}_{H_0}^n / \mathfrak{m}_{H_0}^{n+1}$ graded ring

$$\simeq \mathbb{F}[x_1, \dots, x_{\dim H_0}]$$

$\pi \in \mathrm{Rep}^{adm} H$, $\quad \mathrm{gr}(\pi^\vee) = \bigoplus_{n\geq 0} \mathfrak{m}_{H_0}^n \pi^\vee / \mathfrak{m}_{H_0}^{n+1}\pi^\vee$ is $\mathrm{gr}(\mathbb{F}[\![H_0]\!])$-mod f g.

Björk: $j_{H_0}(\pi^\vee) = \underset{\mathrm{Spec}\,\mathrm{gr}(\mathbb{F}[\![H_0]\!])}{\mathrm{codim}} \mathrm{Supp}\, \mathrm{gr}(\pi^\vee)$

2) $F_v = \mathbb{Q}_p$. $GL_2(F_v) \simeq GL_2(\mathbb{Q}_p)$.

π irred smooth adm rep of $GL_2(\mathbb{Q}_p) \rightsquigarrow \mathrm{GKdim}(\pi) \leq 1$.

3) $\pi \in \mathrm{Rep}^{adm} H$, $\quad \mathrm{GKdim}(\pi) = 0 \iff \dim_{\mathbb{F}} \pi < \infty$.

4) $F_v = \mathbb{Q}_p$ modular curves.

$\mathrm{GKdim}(\pi_v(\bar{r})) = 1$ (2) + local-global Thm, Emerton).

5) $GL_2(F_v) \supset K_n = \mathrm{Id} + p^n M_2(\mathcal{O}_{F_v})$ open pro-p-subgroup.

Emerton-Paskunas:

$\pi \in \mathrm{Rep}^{adm} GL_2(F_v)$, $\quad \delta = \mathrm{GKdim}(\pi)$.

$\exists\, 0 < a < b$, $\forall n$,

$$a \leq \frac{\dim_{\mathbb{F}} \pi^{K_n}}{p^{n\delta}} \leq b$$

6) (BHHMS, Hu-Wang) $GL_2(F_v)$ $\quad F_v/\mathbb{Q}_p$.

$\boxed{\mathrm{GKdim}(\pi_v(\bar{r})) = [F_v : \mathbb{Q}_p] = \dim \text{ of flag variety of } \mathrm{Res}_{F_v/\mathbb{Q}_p} GL_2.}$

* under some genericity hyp on $\bar{r}_w$

* $\bar{r}|_{\mathrm{Gal}(\bar{F}/F(\zeta_p))}$ abs irred.

* other hyp on D and U^v.

Genericity hyp at $v|p$:

$$I_v \subset \mathrm{Gal}(\bar{F}_v/F_v) \xrightarrow{\omega_f} \mathbb{F}^\times$$

$$I_v \twoheadrightarrow \mathcal{O}_{F_v}^\times \twoheadrightarrow \mathbb{F}_{p^f}^\times \overset{L}{\hookrightarrow} \mathbb{F}^\times$$

$\bar{r}_v$ is semisimple reduced

$\omega_f^a \oplus \omega_f^b$ $\quad a - b = \sum_{i=0}^{f-1} (r_i + 1) p^i \mod p^f - 1$

$\forall i \quad 9 \leq r_i \leq p - 12 \Rightarrow p \geq 23.$

3. Other global setups

F tot real D quaternion center F, $D \underset{\mathbb{Q}}{\otimes} \mathbb{R} \simeq \mathbb{H}^d$

$\rightsquigarrow G(\mathbb{R})$ compact mod center at ∞.

$U \subset G(\mathbb{A}_F^\infty)$ / $S_D(U) = \mathcal{C}\left(\underbrace{G(F) \backslash G(\mathbb{A}_F^\infty)/U}, \mathbb{F}\right) \longleftrightarrow H^1_{et}(X_U, \mathbb{F})$.

$\tilde{S}_D(U) = \varinjlim_{U} S_D(U) \circlearrowleft G(\mathbb{A}_F^\infty)$ smooth

$v | p$. F_v unr D split at v. $G(F_v) \simeq GL_2(F_v)$

$U^v \subset G(\mathbb{A}_F^{\infty, v})$

$$\tilde{S}_D(U^v) = \tilde{S}_D^{U^v} \circlearrowleft GL_2(F_v) \text{ smooth + adm.}$$

$\to$ Hecke operators $w \nmid p$ D split at w, $GL_2(\mathcal{O}_{F_w}) \subset U^v$.

$T_w = [U^v (\begin{smallmatrix} \varpi_w & \\ & 1 \end{smallmatrix}) U^v] \curvearrowright \tilde{S}_D(U^v)$.

$\downarrow$

$\mathbb{T}$-module $\mathbb{T} = \mathbb{F}[T_w ; w \text{ as above}]$.

$\hookrightarrow GL_2(F_v)$-equivariant

$\bar{\rho} : Gal(\bar{F}/F) \to GL_2(\mathbb{F})$ abs irred odd, $\bar{\rho}$ unr at w

$\rightsquigarrow m_{\bar{\rho}} \subset \mathbb{T}$ max ideal generated $T_w - Tr(\bar{\rho}(Frob_w))$.

$\Pi_v(\bar{\rho}) = \tilde{S}_D(U^v)[m_{\bar{\rho}}] = Hom_{\mathbb{T}}(\mathbb{T}/m_{\bar{\rho}}, \tilde{S}_D(U^v))$

$\circlearrowleft GL_2(F_v)$

Remark) Shimura curves $\mathbb{T} \curvearrowright \tilde{H}^1(\mathbb{F})^{U^v}$

$\tilde{H}^1(\mathbb{F})^{U^v}[m_{\bar{\rho}}] \underset{Gal \times GL_2(F_v)}{\simeq} \bar{\rho} \otimes \pi(\bar{\rho})^{U^v}$ | Carayol + Boston-Lenstra-Ribet

Unitary groups F^+ tot real F CM ext, quadratic tot imag.

$\rightsquigarrow G_{/F^+}$ $G(R) = \{ GL_n(F \underset{F^+}{\otimes} R) \mid {}^t c(g) g = Id \}$.

$c : F \xrightarrow[/F^+]{\sim} F$ order 2.

$G(F^+ \underset{\mathbb{Q}}{\otimes} \mathbb{R})$ compact

$S_G \longleftrightarrow S_D.$

$v \mid p$, split in F — $v = \tilde{v}\tilde{v}^c$

$\tilde{S}_G(U^v) \circlearrowleft G(F_v^+)$ smooth.

$\phantom{\tilde{S}_G(U^v) \circlearrowleft} \cong GL_n(F_{\tilde{v}})$

$\mathbb{T}$ Hecke algebra. $\mathbb{F}[T_{\tilde{w}}]$ — $\tilde{w} \mid w$, $\tilde{F}$, F^+ — $w \nmid p$. U^v max at w, w split in F.

$\bar{r} : Gal(\bar{F}/F) \to GL_n(\mathbb{F})$ absirred

$\rightsquigarrow m_{\bar{r}} \subset \mathbb{T}$ $\quad T_{\tilde{w}} - Tr(\bar{r}(Frob_{\tilde{w}})).$

$\Pi_v(\bar{r}) = \tilde{S}_G(U^v)[m_{\bar{r}}] \circlearrowleft GL_n(F_{\tilde{v}}).$

$\overset{?}{\cong} \pi(\bar{r}_{\tilde{v}})^{\oplus S}$ for some local mod p correspondence

$\bar{r}_{\tilde{v}} \longmapsto \pi(\bar{r}_{\tilde{v}})$

$Gal(\bar{F}_{\tilde{v}}/F_{\tilde{v}}) \quad GL_n(F_{\tilde{v}})$