THE MOD p REPRESENTATION THEORY OF p-ADIC GROUPS (MAT 1104, WINTER 2012)

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In these exercises, $G = \mathrm{GL}_n(\mathbb{Q}_p)$, $K = \mathrm{GL}_n(\mathbb{Z}_p)$, and E is an algebraically closed field of characteristic p.

Exercise 1 (Maximal compact subgroups of G)**.** A *lattice* in $\mathbb{Q}_p^n$ is a finitely-generated $\mathbb{Z}_p$-submodule of $\mathbb{Q}_p^n$ that generates $\mathbb{Q}_p^n$ as vector space. In particular, it's free of rank n. Note that G acts transitively on the set of lattices in $\mathbb{Q}_p^n$.

(i) Show that $K = \mathrm{Stab}_G(\mathbb{Z}_p^n)$.

(ii) Suppose that K' is a compact subgroup of G. Show that K' stabilises a lattice. (Hint: show that the K'-orbit of $\mathbb{Z}_p^n$ is finite and note that a finite sum of lattices is a lattice.)

(iii) Deduce that every compact subgroup is contained in a maximal compact subgroup and that any maximal compact subgroup is conjugate to K.

Exercise 2. (In this exercise $E = \overline{E}$ can be of any characteristic.) Suppose that π is any irreducible smooth representations of $\mathbb{Q}_p^\times$.

(i) Show that there is an $r \geq 1$ such that $K(r) = 1 + p^r\mathbb{Z}_p$ acts trivially.

(ii) Show that $\mathbb{Z}_p^\times$ acts on π via a smooth character $\mathbb{Z}_p^\times \to E^\times$.

(iii) By twisting we can assume that $K = \mathbb{Z}_p^\times$ acts trivially, so π is an irreducible representation of $G/K \cong \mathbb{Z}$. Show that is π is one-dimensional.

Exercise 3 (Modular representations of finite groups)**.** Suppose Γ is a finite group. Say that $\gamma \in \Gamma$ is *p-regular* (resp. *p-singular*) if the order of γ is prime to p (resp. a power of p). The aim of this exercise is to show that the number of irreducible Γ-representations over E is at most the number of p-regular conjugacy classes. (In fact, equality holds.) This will show that in class we constructed *all* irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$.

(i) Show that every element $\gamma \in \Gamma$ can be uniquely written as $\gamma_r\gamma_s = \gamma_s\gamma_r$, where γ_r is p-regular and γ_s is p-singular.

(ii) Suppose that $g \in \mathrm{GL}_d(E)$ is of finite order. Show that g is *p-regular* (resp. *p-singular*) iff g is diagonalisable (resp. unipotent).

(iii) Suppose that ρ is an irreducible Γ-representation. Show that $\mathrm{tr}\,\rho : \Gamma \to E$ is a class function that is determined by its restriction to the set of p-regular elements. (Hint: show that $\mathrm{tr}\,\rho(\gamma) = \mathrm{tr}\,\rho(\gamma_r)$.)

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(iv) Suppose that $\rho_1,\dots,\rho_r$ are non-isomorphic irreducible Γ-representations. Show that $\operatorname{tr}\rho_i : \Gamma \to E$ are linearly independent. (Hint: use the result of Burnside that the group ring $E[\Gamma]$ surjects onto $\prod \operatorname{End}_E(\rho_i)$. Burnside's result holds whenever E is algebraically closed and ρ_i are non-isomorphic and irreducible. It's a consequence of the Artin-Wedderburn classification of semisimple rings.)

(v) Deduce the result.

Exercise 4 (Modular representations of $\mathrm{GL}_2(\mathbb{F}_q)$)**.** Say $q = p^f$. Throughout, fix an embedding $\mathbb{F}_q \to E$, so $\Gamma := \mathrm{GL}_2(\mathbb{F}_q)$ acts on E^2. Let $\phi : \Gamma \to \Gamma$ denote the homomorphism that sends a matrix (a_{ij}) to (a_{ij}^p). If V is a Γ-representation, let $V^{(i)}$ denote the representation $\Gamma \xrightarrow{\phi^i} \Gamma \to \mathrm{GL}(V)$. (So $V^{(f)} \cong V$.) The aim of this exercise is to show that the irreducible Γ-representations are given by:

$$\bigotimes_{i=0}^{f-1} (\operatorname{Sym}^{a_i} E^2)^{(i)} \otimes \det{}^b, \tag{0.1}$$

where $0 \le a_i \le p-1$ and $0 \le b < q-1$. Write $a := \sum a_i p^i$.

(i) To show irreducibility, we may suppose $b = 0$. Show that the representation above is isomorphic to the subrepresentation of $\operatorname{Sym}^a E^2$ (thought of as homogeneous polynomials in X, Y of degree a) that has basis $X^m Y^{a-m}$, where $m = \sum m_i p^i$ and $0 \le m_i \le a_i$ for all i.

(ii) As in class show that the $\begin{pmatrix} 1 & \mathbb{F}_q \\ & 1 \end{pmatrix}$-invariant vectors are spanned by X^a.

(iii) Show that X^a generates the representation. (As in class, use a Vandermonde determinant.)

(iv) Deduce that the representations in (0.1) are irreducible and non-isomorphic.

(v) Using the previous exercise show that we have found all irreducible Γ-representations.

Exercise 5. Recall that $F(a,b) = \operatorname{Sym}^{a-b}(E^2) \otimes \det^b$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$ when $a - b \le p-1$.

(i) Show that $F(a,b)^* \cong F(-b,-a)$. (Hint: for $k < p$ the usual natural pairing shows that $(\operatorname{Sym}^k \sigma)^* \cong \operatorname{Sym}^k(\sigma^*)$, so can reduce to $a = 1$, $b = 0$. Show that for any 2-dimensional representation σ of any group that $\sigma^* \cong \sigma \otimes \det^{-1}$.)

(ii) Suppose Γ is a finite group and V a Γ-representation. Show that $(V^*)^\Gamma \cong (V_\Gamma)^*$. Use this to compute $F(a,b)_{\overline{U}(\mathbb{F}_p)}$ in a different way than we did in class.

Exercise 6 (Compact and parabolic inductions)**.** Suppose that $n = 2$. Recall that for any weight V in a principal series $\operatorname{Ind}_{\overline{B}}^{G} \chi$ we constructed a natural injective map

$$(\text{c-Ind}_K^G V)[T_1^{-1}] \to \operatorname{Ind}_{\overline{B}}^{G}(\text{c-Ind}_{T(\mathbb{Z}_p)}^{T} V_{\overline{U}(\mathbb{F}_p)}) \tag{0.2}$$

that is $\mathcal{H}_G(V)[G]$-linear. We showed that it is surjective when $\dim V > 1$. Show that it fails to be surjective when $\dim V = 1$. (Pick a smooth character $\chi : \mathbb{Q}_p^\times \to E^\times$ such that $\chi \circ \det|_{T(\mathbb{Z}_p)} = V_{\overline{U}(\mathbb{F}_p)}$ and compose (0.2) with the natural surjection to $\mathrm{Ind}_{\overline{B}}^G(\chi \circ \det)$. Show that the image of $(\text{c-Ind}_K^G V)[T_1^{-1}]$ lands in the one-dimensional subrepresentation of $\mathrm{Ind}_{\overline{B}}^G(\chi \circ \det)$.)

Exercise 7 (Steinberg representation)**.** Suppose that $n = 2$. Recall that $\mathrm{St} = C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)/1$, where we identified $\overline{B}\backslash G$ with $\mathbb{P}^1(\mathbb{Q}_p)$ via the first row. The goal of this exercise is to show that $\dim \mathrm{St}^{I(1)} = 1$. This completes the proof of irreducibility of St given in class, and also shows that St is admissible.

(i) Show that $\dim C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)^{I(1)} = 2$. (For example, show that $\overline{B}\backslash G/I(1)$ has two elements by the Cartan and the Bruhat decompositions.)

(ii) It remains to show that the map $C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)^{I(1)} \to \mathrm{St}^{I(1)}$ is surjective. Suppose that $f \in C_c^\infty(\mathbb{P}^1(\mathbb{Q}_p), E)$ maps to an element of $\mathrm{St}^{I(1)}$. Show that the stabiliser of f in $I(1)$ contains any element having a fixed point on $\mathbb{P}^1(\mathbb{Q}_p)$.

(iii) Complete the proof by showing that $I(1) = \begin{pmatrix} 1 & \\ p\mathbb{Z}_p & 1 \end{pmatrix} \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ & \mathbb{Z}_p^\times \end{pmatrix}$, noting that the matrices in this product fix $(1 : 0)$, resp. $(0 : 1)$, in $\mathbb{P}^1(\mathbb{Q}_p)$.

Exercise 8 (Steinberg representation II)**.** Again, $n = 2$. The goal of this exercises is to give an alternative proof of irreducibility of St, by showing that St is irreducible even as B-representation.

(i) Show that the "extension by zero" map $C_c^\infty(\mathbb{Q}_p, E) \to \mathrm{St}$ is an isomorphism of B-representations. Recall that T acts on the left by scaling and U by translations.

(ii) Suppose that π is any nonzero B-subrepresentation of $C_c^\infty(\mathbb{Q}_p, E)$. Show that $\pi \cap C_c^\infty(\mathbb{Z}_p, E) \neq 0$.

(iii) Use the p-groups lemma to show that π contains the characteristic function $1_{\mathbb{Z}_p}$.

(iv) Use scaling and translation to show that $\pi = C_c^\infty(\mathbb{Q}_p, E)$.

Exercise 9 (Schur's lemma)**.** Suppose that E is *uncountable* (of arbitrary characteristic). Let π be an irreducible smooth G-representation and suppose that $f : \pi \to \pi$ is a non-zero G-linear map having no eigenvector.

(i) Show that $\dim_E \pi$ is countable. (Hint: one way to do this uses the Iwasawa decomposition, another way uses lattices as in Exercise 1.)

(ii) Show that if $P \in E[T]$ is a non-zero polynomial, then $P(f) : \pi \to \pi$ is an isomorphism.

(iii) Fix $v \in \pi$ non-zero. Note that the elements $\{(f - \lambda)^{-1}v : \lambda \in E\}$ are linearly dependent, and deduce a contradiction.

(iv) Prove that $\mathrm{End}_G(\pi) = E$. In particular, π has a central character.

Exercise 10 (Finite-dimensional irreducible representations)**.** Suppose that π is a finite-dimensional irreducible smooth G-representation.

(i) Show that there is an open normal subgroup of G that acts trivially.
(ii) Show that U and $\overline{U}$ both act trivially. (Use the torus action.)
(iii) Deduce that there is a smooth character $\chi : \mathbb{Q}_p^\times \to E^\times$ such that $\pi \cong \chi \circ \det$. (Hint: it's known that U and $\overline{U}$ generate $\mathrm{SL}_n(\mathbb{Q}_p)$. This is in fact true over any field.)

Exercise 11. Recall that in the proof of the Satake isomorphism we crucially used a certain compatibility relation between Cartan and Iwasawa decompositions. Let $\overline{U}$ denote the unipotent radical of the lower-triangular Borel subgroup. Let $\Lambda_- = \{\lambda \in \Lambda = \mathbb{Z}^n : \lambda_1 \le \cdots \le \lambda_n\}$. For any $\mu \in \Lambda$ let $t_\mu \in T$ be defined as the diagonal matrix $\mathrm{diag}(p^{\mu_1}, \dots, p^{\mu_n})$. For all $\lambda \in \Lambda_-$ and $\mu \in \Lambda$ we want to show that $\overline{U} t_\mu \cap K t_\lambda K \neq \varnothing$ implies that $\mu \ge \lambda$, i.e., that $\sum_{i=1}^r \mu_i \ge \sum_{i=1}^r \lambda_i$ for all r, with equality when $r = n$.

(i) Show that $\sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i$. [This would also follow from the general argument below.]
(ii) Show that $\mu_1 \ge \lambda_1$.
(iii) Now reduce the general case to the previous case: let $V = E^n$ be the vector space on which G acts. We have a homomorphism $G = \mathrm{GL}_E(V) \to \mathrm{GL}_E(\bigwedge^r V)$, letting G act in the natural way on $\bigwedge^r V$. The standard basis $(e_i)_{i=1}^n$ of V gives rise to the basis $e_{i_1} \wedge \cdots \wedge e_{i_r}$ with $1 \le i_1 < \cdots < i_r \le n$. Apply this homomorphism to $\overline{U} t_\mu \cap K t_\lambda K \neq \varnothing$ and apply part (ii) to deduce $\sum_{i=1}^r \mu_i \ge \sum_{i=1}^r \lambda_i$.
(iv) Use the same argument to show that $\overline{U} t_\lambda \cap K t_\lambda K = (\overline{U} \cap K) t_\lambda$. (It helps to order the basis of $\bigwedge^r V$ by the lexicographic order.)

[This is similar to Satake's argument in his 1963 paper. He notes, however, that for the purpose of establishing his isomorphism it suffices to show that $\mu \ge_\ell \lambda$ in the *lexicographic* order $\ge_\ell$ (the point is that if $\lambda \in \Lambda_-$ is fixed, then there are only finitely many $\mu \in \Lambda_-$ with $\sum \mu_i = \sum \lambda_i$ and $\mu \ge_\ell \lambda$), which is a little easier.]

Exercise 12 (Explicit Satake transform for GL_2)**.** Suppose that $n = 2$. Suppose that V is a weight of K. Recall that, with the notation of the previous exercise, for $\lambda \in \Lambda_-$ we denote by $T_\lambda \in \mathcal{H}_G(V)$ the unique element of support $K t_\lambda K$ such that $T_\lambda(t_\lambda) \in \mathrm{End}_E(V)$ is a linear projection. Recall also that for $\lambda \in \Lambda$ we denote by $\tau_\lambda \in \mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$ the unique element of support $(T \cap K) t_\lambda$ such that $\tau_\lambda(t_\lambda) = 1$.

For $\lambda \in \Lambda_-$ show that $\mathcal{S}_G(T_\lambda) = \tau_\lambda$ if $\dim_E V > 1$ or if $\lambda_1 - \lambda_2 \ge -1$, and $\mathcal{S}_G(T_\lambda) = \tau_\lambda - \tau_{\lambda+(1,-1)}$ otherwise. Use this to express $T_{0,1} T_\lambda$ in terms of the T_μ, and compare with the formulae of Barthel–Livné in [BL94], Proposition 8. [It's also possible to reverse the argument and first compute $T_{0,1} T_\lambda$, which inductively gives a formula for $\mathcal{S}_G(T_\lambda)$. There's also a much more general formula for (the inverse of) $\mathcal{S}_G$, see [Her11], Proposition 5.1.]

Exercise 13 (Explicit Satake transform for GL_2, part II)**.** For $b \in \mathbb{Z}$ consider the weights $V = F(b,b) = \det^b$ and $V' = F(b+p-1,b)$. Consider Hecke operators $\varphi_+ \in \mathcal{H}_G(V,V')$ and $\varphi_- \in \mathcal{H}_G(V',V)$ whose support is $K\begin{pmatrix} 1 & \\ & p \end{pmatrix}K$. (We know that these exist and are unique up to nonzero scalar.) Fix an isomorphism $V_{\overline{U}(\mathbb{F}_p)} \xrightarrow{\sim} (V')_{\overline{U}(\mathbb{F}_p)}$, so that we can identify $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)}, (V')_{\overline{U}(\mathbb{F}_p)})$, $\mathcal{H}_T((V')_{\overline{U}(\mathbb{F}_p)}, V_{\overline{U}(\mathbb{F}_p)})$, $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$.

(i) Show that $\mathcal{S}_G(\varphi_+) = \tau_{0,1}$ and $\mathcal{S}_G(\varphi_-) = \tau_{0,1} - \tau_{1,0}$ in $\mathcal{H}_T(V_{\overline{U}(\mathbb{F}_p)})$ (up to nonzero scalar).

(ii) Deduce that $\varphi_+ * \varphi_- = \varphi_- * \varphi_+ = T_1^2 - T_2$ (the latter up to nonzero scalar) in $\mathcal{H}_G(V) \cong \mathcal{H}_G(V')$, as we stated earlier.

Exercise 14. In class we proved the Satake isomorphism for $G = \mathrm{GL}_n(\mathbb{Q}_p)$. The purpose of this exercise is to show that it also works for standard Levi subgroups of G. Suppose that $M \cong \mathrm{GL}_{n_1}(\mathbb{Q}_p) \times \cdots \times \mathrm{GL}_{n_r}(\mathbb{Q}_p)$ (in this order). First, define the Satake transform by the Yoneda lemma just as in the GL_n-case. It is an algebra homomorphism $\mathcal{S}_M : \mathcal{H}_M(V) \to \mathcal{H}_T(V_{(\overline{U}\cap M)(\mathbb{F}_p)})$ for V a weight of $M \cap K$ (which is nothing but a tensor products of weights of $\mathrm{GL}_{n_i}(\mathbb{Z}_p)$). Show that its image consists of those functions that are supported on $T^{-,M} = \{\mathrm{diag}(t_1, \dots, t_n) : \mathrm{ord}(t_1) \le \cdots \le \mathrm{ord}(t_{n_1}),\ \mathrm{ord}(t_{n_1+1}) \le \cdots \le \mathrm{ord}(t_{n_1+n_2}),\ \dots\}$.

[This is a somewhat lengthy exercise, but each step of the argument generalises from the GL_n-case.]

Exercise 15 (Transitivity of parabolic induction)**.** Suppose that $P = M \ltimes N$ and $Q = L \ltimes N'$ are standard parabolic subgroups of G such that $P \subset Q$. (In particular, $M \subset L$ and $N \supset N'$.) Prove that for smooth M-representations σ, we have a natural isomorphism

$$\theta : \mathrm{Ind}_{\overline{P}}^{G} \sigma \cong \mathrm{Ind}_{\overline{Q}}^{G} \left(\mathrm{Ind}_{\overline{P}\cap L}^{L} \sigma \right),$$

where, as usual, we consider σ as $\overline{P}$-representation via the natural projection $\overline{P} \twoheadrightarrow M$ and similarly we consider the induced representation inside parentheses as $\overline{Q}$-representation.

(Hint: first note that $\overline{P} \cap L = M \ltimes (\overline{N} \cap L)$. The isomorphism can be described by $\theta(f)(g)(l) = f(lg)$ and $\theta^{-1}(F)(g) = F(g)(1)$.)

Exercise 16 (Generalised Steinberg representations)**.** In class I explained without too many details that the generalised Steinberg representations

$$\mathrm{Sp}_P = \frac{\mathrm{Ind}_{\overline{P}}^{G} 1}{\sum_{Q \supsetneq P} \mathrm{Ind}_{\overline{Q}}^{G} 1},$$

for standard parabolic subgroups P are irreducible and are pairwise non-isomorphic [GK]. Let n_P denote the number of GL-blocks of the Levi of P. Let $\pi_i := \sum \mathrm{Ind}_{\overline{P}}^{G} 1$, where the sum is over all standard parabolics with $n_P = i$. Then π_i is an increasing filtration of $\mathrm{Ind}_{\overline{B}}^{G}(1)$.

Show by induction on i that the irreducible constituents of π_i are the Sp_P with $n_P \leq i$, each occurring with multiplicity one. Deduce in particular that the irreducible constituents of $\mathrm{Ind}_B^G(1)$ are all the Sp_P, each occurring with multiplicity one. [I thank E. Große-Klönne for this suggestion.]

References

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