Math 470-1, Fall 2009

Graduate Algebra
Homework 3

1. Suppose that K is algebraically closed and that σ is a ring automorphism of K. Show that any finite extension of K^σ (the fixed field of σ) is Galois with cyclic Galois group.

2. Let K be the normal closure of $\mathbb{Q}(\sqrt{1+\sqrt{3}})/\mathbb{Q}$.

   (a) Find $\mathrm{Gal}(K/\mathbb{Q})$. (Give a carefully justification why the degree is what you think it is!)

   (b) Let H be the unique subgroup of $\mathrm{Gal}(K/\mathbb{Q})$ that is cyclic of order 4. Determine K^H.

   (c) Find the diagram of intermediate fields (w.r.t. inclusion).

3. Find the diagram of intermediate fields for $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

4. Let K/k be a finite Galois field extension, and write $G = \mathrm{Gal}(K/k)$. Let E be a finite extension of k, and write $\Sigma := \mathrm{Hom}_{k-alg}(E, K)$. Note that Σ is naturally a transitive G-set (G acts on Σ by composition). Suppose that $\Sigma \neq \varnothing$.

   Evaluation of homomorphisms gives a pairing $E \times \Sigma \to K$ which is compatible with the G-actions on each of Σ and K, and hence induces a homomorphism of k-algebras $E \to \mathrm{Map}_G(\Sigma, L)$. (The target denotes the set of all G-equivariant maps. Convince yourself that it is naturally a commutative k-algebra by pointwise addition/multiplication.) Prove that the map is an isomorphism of k-algebras.

   [Recall that a k-algebra is a ring A together with a ring homomorphism $k \to A$. So A is a ring that is also a k-vector space, and the two structures are compatible with each other. A homomorphism of k-algebras $A \to B$ is a ring homomorphism such that the two maps $k \to B$ agree.]

5. Let K/k and G be as in the previous question and let Σ now be some given transitive G-set. Write $E := \mathrm{Map}_G(\Sigma, K)$; as observed before, it is naturally a commutative k-algebra.

   (a) Prove that E is a field.

   (b) Evaluation of maps gives a pairing $E \times \Sigma \to K$ which is compatible with the G-actions on each of Σ and K, and hence induces a G-equivariant map $\Sigma \to \mathrm{Hom}_{k-alg}(E, K)$. Prove that this map is an isomorphism of G-sets.

6. In solving the previous two questions, you probably used the fundamental theorem of Galois theory (i.e., the bijection between subgroups and intermediate fields). Conversely, use the conclusions of the previous two questions to derive the fundamental theorem of Galois theory.

7. Let $a \in \mathbb{Q}^\times$.

   (a) Let p be a prime, and suppose that a is not a perfect p-th power. Let L be the splitting field of the polynomial $x^p - a$ in $\mathbb{C}$. Describe the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ as explicitly as possible.

   (b) Keep the hypotheses of (a). For any $n > 0$, let L_n denote the splitting field of the polynomial $x^{p^n} - a$ in $\mathbb{C}$. Describe the Galois group $\mathrm{Gal}(L_n/\mathbb{Q})$ as explicitly as possible.

   (c) Keep the hypotheses of (a) and (b). Define $L_\infty := \bigcup_{n \geq 0} L_n$. Describe $\mathrm{Gal}(L/\mathbb{Q})$ as explicitly as possible.

   In (a) and (b) you should justify why the degree of the extension is what you think it is. This is maybe tricky in (b). The hint is to use what you found in (a): L is a subextension of L_n, so one Galois group surjects on the other...

8. (a) Find all closed subgroups of $\mathbb{Z}_p$. It may help to show using the p-adic norm that if x is in a closed subgroup H, then $x\mathbb{Z}_p \subset H$.

   (b) Which of the closed subgroups are open?

   (c) Find all intermediate fields in $K/\mathbb{F}_q$, where q is a prime power and $K = \bigcup_{i \geq 0} \mathbb{F}_{q^{\ell^i}}$ (ℓ prime). You could do this using (a) or without it.