Math 470-1, Fall 2009

Graduate Algebra
Homework 2

1. Suppose that k is a field and that R is a finite-dimensional commutative k-algebra (i.e., R is a commutative ring together with a ring homomorphism $k \to R$ such that R is finite-dimensional as k-vector space). If R is a domain show that R is a field.

2. Suppose that α is algebraic over a field k. Explain how to compute α^{-1} as element of $k[\alpha]$ (i.e., as polynomial in α over k). [There are several methods.]

   Apply your method to compute $(1+\sqrt{2}+\sqrt{3})^{-1}$.

3. Lang, problems 2, 7, 11d, 20, 26 (Chapter V).

   [Note for 7: EF denotes the smallest extension of K containing both E and F. Hint for 20b: use Gauß' lemma, which is valid for any UFD: I mean the version of Gauß' lemma that says that a polynomial in $R[x]$ is irreducible over R, it is irreducible over $Q(R)$.]

4. Find the degree of $\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$. Is it a normal extension?

5. Suppose that K/k is an algebraic extension. A *normal closure* of K/k is an extension L/K such that (i) L/k is normal and (ii) no proper subfield of L that contains K is normal over k (i.e., L is minimal w.r.t. (i)).

   (a) Show that K/k has a normal closure and that any two normal closures are isomorphic.

   (b) If K/k is finite then any normal closure is also finite.

   (c) If K/k is separable, show that the normal closure is separable (hence Galois) over k.

   (d) Find a normal closure of $\mathbb{Q}(\sqrt[5]{3})/\mathbb{Q}$. What is its degree over $\mathbb{Q}$?

6. Suppose that k is a field of characteristic $p > 0$. Let $L = k(s^{1/p}, t^{1/p})$ and $K = k(s,t)$. (Here s, t do not satisfy any polynomial relation, i.e., K is the field of fractions of $k[s,t]$.) Consider the field extension L/K. What is its degree? Show that there does not exist an element $\alpha \in L$ such that $L = K(\alpha)$.

7. Suppose that K/k is an algebraic extension and $\alpha_i \in K$ $(i \in I)$ such that $K = k(\alpha_i : i \in I)$. If α_i is separable over k for all i, then K/k is separable.