Math 470-3, Spring 2010

Graduate Algebra
Homework 3

1. Suppose that k is a field. Find the associated primes of the $k[x,y,z]$-module $k[x,y,z]/(x^2, xy^2, yz)$. Write each one as $\mathrm{Ann}(m)$ for some m in the module.

2. Suppose that $\mathbb{T}$ is a ring that is finitely generated as $\mathbb{Z}$-module. Suppose that M is a finite and faithful $\mathbb{T}$-module. (Faithful means: for all $T \neq 0$ in $\mathbb{T}$ there is an $m \in M$ with $Tm \neq 0$.) Suppose that $\mathfrak{m} \subset \mathbb{T}$ is a maximal ideal. Show that there is a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that $M[\mathfrak{p}] \neq 0$, i.e., there is an $f \in M$, $f \neq 0$ such that $Tf = 0$ for all $T \in \mathfrak{p}$.

3. Suppose A is a ring that is complete with respect to the I-adic topology. Show that $I \subset \mathrm{rad}\, A$.

4. (a) Let A be a noetherian ring, $\mathfrak{p}$ a prime ideal such that $\mathrm{ht}(\mathfrak{p}) \geq 2$. Prove that $\mathfrak{p}$ contains infinitely many prime ideals of height 1. (Hint: deduce a contradiction to Krull's principal ideal theorem.)

   (b) Let $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of a noetherian ring A. Prove that if there is a prime ideal strictly between $\mathfrak{p}$ and $\mathfrak{q}$, then there are infinitely many such prime ideals.

5. Let A be a dvr. Show that $A[\![x]\!]$ is a regular local ring of dimension 2. You may assume that this ring is noetherian. (As I mentioned in class, this follows by a similar argument as Hilbert's basis theorem.) (Hint: an earlier homework problem might be useful.)

6. Suppose that k is a field. Show that $k[\![x]\!]$ is a dvr. Describe the corresponding discrete valuation v of the fraction field and give a uniformiser.

Other exercises (not required):

1. Find the associated primes of $k[x,y,u,v]/(xy, uv, xu+yv)$. (This is lengthier...)

2. Suppose that $(A, \mathfrak{m})$ is a noetherian local ring and suppose that $x \in \mathfrak{m}$ is a non-zero-divisor. If A/xA is a domain, show that A is a domain.