Math 470-3, Spring 2010

Graduate Algebra
Homework 2

In problems 2–5, k denotes an algebraically closed field.

1. Suppose A is noetherian and $I \subset A$ an ideal. Show that $(\sqrt{I})^n \subset I$ for some $n \geq 0$. In particular, the nilradical is a nilpotent ideal.

2. Suppose that $I \subset k[x_1, \ldots, x_n]$ is an ideal. Show that $\sqrt{I} = \bigcap_{\mathfrak{m} \text{ max.}, \mathfrak{m} \supset I} \mathfrak{m}$. What is $\operatorname{rad} k[x_1, \ldots, x_n]$?

   In particular, in $k[x_1, \ldots, x_n]$ every prime ideal is an intersection of some family of maximal ideals. Show that this doesn't hold for general rings.

3. Suppose that $X \subset k^n$ is an arbitrary subset. Show that $V(I(X)) = \overline{X}$ (the closure of X in the Zariski topology).

4. Suppose $V \subset k^n$ is a variety with ring of regular functions $k[V] = k[x_1, \ldots, x_n]/I(V)$. Let $P \in V$ be a point and let $\mathfrak{m} \subset k[V]$ be the corresponding maximal ideal. The aim of this problem is to show that the localisation $k[V]_{\mathfrak{m}}$ can be interpreted as the ring of germs of regular functions at the point P.

   Recall that V carries the Zariski topology (induced from the one on k^n). Define the ring of regular functions on an open subset $U \subset V$ as follows: $\mathcal{O}_V(U)$ consists of all functions $f : U \to k$ such that for each $x \in U$ there is an open neighbourhood $x \in U' \subset U$ and functions p, $q \in k[V]$ with q not having any zero on U' such that $f = p/q$ on U'.

   The ring of germs of regular functions at P is $\mathcal{O}_{V,P}$. Its elements are pairs (U, f) consisting of an open subset $P \in U \subset V$ and $f \in \mathcal{O}_V(U)$, modulo the equivalence relation that $(U, f) \sim (U', f')$ whenever there is a an open neighbourhood $P \in U'' \subset U \cap U'$ such that $f = f'$ on U''. (It's easy to check that this is a k-algebra.)

   You may assume that V is irreducible. This isn't necessary, but makes the argument a little easier.

   (a) Show that there is a well-defined k-algebra homomorphism $\mathcal{O}_V(U) \to k[V]_{\mathfrak{m}}$ whenever $P \in U$. (This is a bit subtle, do this very carefully!)

   (b) Find a k-algebra homomorphism $\mathcal{O}_{V,P} \to k[V]_{\mathfrak{m}}$. (This should now be straightforward.)

   (c) Show that the map in (b) is an isomorphism.

5. (a) Suppose that $f \in k[x_1, \ldots, x_n]$ is non-zero. Determine the irreducible components of $V((f))$.

   (b) Find the irreducible components of the variety $V = V((x^2 - yz, x - xz)) \subset k^3$.

6. Compute the normalisation of the integral domain $k[x, y]/(y^2 - x^2(x+1))$. Here k is any field.
7. Atiyah-Macdonald: §6: 3, 4
8. Atiyah-Macdonald: §8: 3
9. Atiyah-Macdonald: §5: 1, 10(i)
10. Atiyah-Macdonald: §1: 21, 22
11. Atiyah-Macdonald: §3: 21. As a corollary deduce that if $A \to B$ is finite then the map on Specs has finite fibres. (Hint: where did we see rings with only finitely many prime ideals before?)

Other exercises that are useful (but not required):

- Can you do #4 without assuming that V is irreducible?
- In #4 show that $\mathcal{O}_V(D(f))$ is naturally isomorphic to the localisation $k[V]_f$, for any $f \in k[V]$.