Math 470-3, Spring 2010

Graduate Algebra
Homework 1

1. Suppose that $A \to B$ is a ring homomorphism.
   (a) Show that the inverse image of a prime ideal is prime.
   (b) Show that the inverse image of a maximal ideal needn't be maximal.
   (c) Suppose that k is a field and $A \to B$ a map of k-algebras. If B is of finite type over k show that the inverse image of a maximal ideal is maximal. (Hint: Zariski's lemma.)

2. (a) Suppose that M is an A-module and $I \subset A$ an ideal. Show that $M \otimes_A A/I \cong M/IM$. (Hint: use that $\otimes_A$ is right exact.) Use this to simplify $A/I \otimes_A A/J$.
   (b) Let k be a field. Show that $k[x] \otimes_k k[y] \cong k[x, y]$ as k-algebras. (Hint: you could show that they satisfy the same universal property.) What do x, $y \in k[x, y]$ correspond to on the left-hand side, under your isomorphism?

3. Suppose that $I \subset A$ is an ideal. Let S be a multiplicative subset of A and let $\overline{S}$ be its image in A/I (again a multiplicative subset). Show that $S^{-1}A/S^{-1}I \cong \overline{S}^{-1}(A/I)$.

4. (a) Suppose that $A \to B$ is a ring homomorphism and that M and N are A-modules. Show that $B \otimes_A (M \otimes_A N) \cong (B \otimes_A M) \otimes_B (B \otimes_A N)$ as B-modules.
   (b) You probably used a series of isomorphisms to establish (a). Trace them through to describe both the isomorphism and its inverse explicitly (i.e., where does it send $b \otimes (m \otimes n)$ and where does its inverse send $(b_1 \otimes m) \otimes (b_2 \otimes n)$?).
   (c) Now suppose that S is a multiplicative subset of A and show that $S^{-1}(M \otimes N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N$. Again explicitly describe the isomorphism you obtained and its inverse.

5. Atiyah-Macdonald: §1: 5 (don't worry about the converse in (ii) and see p. 9 for "contraction"), 12

6. Atiyah-Macdonald: §2: 8, 9, 11 (first two parts), 12

7. Atiyah-Macdonald: §3: 1, 5, 12 (ignore the hint for (iv), it's much easier if you think about $K \otimes_A M$ in a different way!), 13

Other recommended exercises (but not required): §2: 14–20 about direct limits. Just like inverse limits, these are useful in various contexts.