MAT 347
Orbits and Stabilizers, Cyclic groups
September 24, 2019

Orbits and Stabilizers

Definitions. Let G be a group acting on a set A.

- Given $a \in A$, we define the *stabilizer* of a as the set

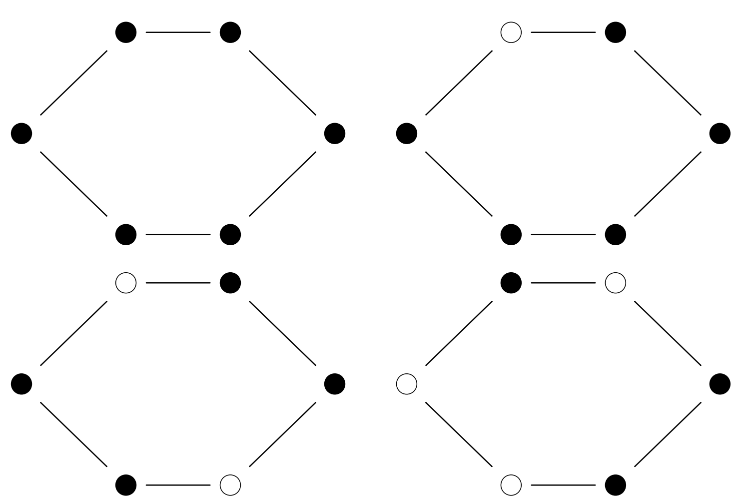
$$\text{Stab}(a) := \{g \in G \mid g \cdot a = a\} \subseteq G.$$

- Given $a \in A$ we define the *orbit* of a as the set

$$Ga := \{g \cdot a \mid g \in G\} \subseteq A.$$

1. Verify that $\text{Stab}(a)$ is a subgroup of G.
2. Say we want to count how many *different* necklaces we can build with 6 stones each, if we have stones of two different colours. Define a *diagram* to be any way of colouring each of the six vertices of a hexagon with black or white. Notice that $|A| = 64$. Show that D_{12} acts on A, and that the number of orbits of this action equals the number of different necklaces.

   Note: This shows that the problem of counting the number of orbits of an action is an interesting problem in combinatorics.
3. Regarding the previous question, consider the following diagrams:



For each one of them, compute the size of its orbit and the size of its stabilizer. Make a conjecture or a formula that relates these two numbers for an arbitrary element in an arbitrary action. Then prove it.

4. If a group G acts on a set A and Ga, Gb are any two orbits, what can we say about how Ga and Gb relate to each other? (For example, what happens if the two orbits have any element in common?)

Cyclic groups

Definition: A group is *cyclic* if it has a generating set with a single element. In other words, a group G is cyclic if there exists $a \in G$ such that

$$G := \{a^n \mid n \in \mathbb{Z}\}.$$

When this happens, we write $G = \langle a \rangle$.

5. If G is a cyclic group generated by a, what is the relation between $|G|$ and $|a|$? Remember that $|G|$ is the order of G, namely its cardinality. On the other hand, $|a|$ is the order of the element a, which has a different definition.

6. True of False? A group G is cyclic if and only if it contains an element whose order equals $|G|$.

7. Prove that two cyclic groups are isomorphic if and only if they have the same order.

Because of Problem 7, given any positive integer n, we define C_n to be *the* cyclic group of order n. We normally use a multiplicative notation for it. A presentation of C_n would be

$$C_n := \langle a \mid a^n = 1 \rangle.$$

Notice that $C_n \cong \mathbb{Z}/n\mathbb{Z}$ (we still use additive notation for the latter!).

On the other hand, we normally think of $(\mathbb{Z}, +)$ as *the* cyclic group of infinite order, with additive notation.

8. Let G be a cyclic group generated by a. What are all the generators of G? (Here I am asking, which other elements of G generate G?) How many of them are there? You may want to think of the finite and infinite cases separately. If you do not know how to start, consider the specific cases C_6 and C_{12} first.