MAT 347
Group actions
September 23, 2019

Actions

Definition. Let G be a group and let A be a set. An *action* of G on A is a map

$$\begin{aligned} G \times A &\longrightarrow A \\ (g, a) &\mapsto g \cdot a \end{aligned}$$

that satisfies the following two properties:

- $1 \cdot a = a$ for all $a \in A$,
- $g \cdot (h \cdot a) = (gh) \cdot a$ for all $a \in A$, $g, h \in G$.

1. Which ones of the following are actions?
   (a) For any set A, $G = S_A$ and the map is the natural one.
   (b) G is any group, $A = G$ as a set, and the map is $g \cdot a := ga$.
   (c) G is any group, $A = G$ as a set, and the map is $g \cdot a := ag$.
   (d) G is any group, $A = G$ as a set, and the map is $g \cdot a := gag^{-1}$.
   (e) $G = D_{2n}$, A is the set of vertices of a regular n-gon, and the map is the natural one.
   (f) $G = D_{2n}$, A is the set of diagonals of a regular n-gon, and the map is the natural one.
   (g) For any set B, $G = S_B$ and A is the set of subsets of B, and the map is ...
2. We say that an action is *transitive* if for all $a, b \in A$, there exists $g \in G$ such that $g \cdot a = b$. Which of the actions in the previous problem are transitive?
3. Assume we have an action of the group G on the set A. For each $g \in G$, let us define a map $\phi_g : A \to A$ by the equation $\phi_g(a) := g \cdot a$. Show that ϕ_g is a bijection. This defines a map $\phi : G \to S_A$ by the equation $\phi(g) := \phi_g$. Show that ϕ is a group homomorphism.
4. Conversely, show that every group homomorphism $G \to S_A$ comes from an action of G on A. In other words, there is a natural one-to-one correspondence between actions of G on A and group homomorphisms from G to S_A. This is why some authors define an action as a group homomorphism $G \to S_A$ instead.