MAT 347
The Galois group
March 3, 2020

Note that we already covered questions 1–3 in class.

Definition 1 *Let K/F be a field extension. The Galois group of K over F is defined as*

$$\mathrm{Gal}(K/F) = \{\varphi : K \to K | \varphi \textit{ is an automorphism and } \varphi(a) = a \textit{ for all } a \in F\}$$

(which is a group under composition).

1. Show that $\mathrm{Aut}(K) = \mathrm{Gal}(K/F)$, if F is the prime subfield of K.

2. Show that $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \tau\}$, where τ is complex conjugation. (Hint: consider $\varphi(a + bi)$ for $\varphi \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. What do you know about $\varphi(i)$?)

3. Suppose that $\varphi \in \mathrm{Gal}(K/F)$ and $f(x) \in F[x]$.

   (a) If $\alpha \in K$ is a root of $f(x)$, then so is $\varphi(\alpha)$.

   (b) Show that $\alpha \in K$ is algebraic over F iff $\varphi(\alpha)$ is algebraic over F, and that in that case they have the same minimal polynomial over F.

   (c) Deduce that there is an action of $\mathrm{Gal}(K/F)$ on the set of roots of $f(x)$ in K.

4. Suppose $K = F(\alpha)$ for some algebraic $\alpha \in K$, and α has minimal polynomial $f(x)$ over F. Then $|\mathrm{Gal}(K/F)|$ = number of roots of $f(x)$ in K. (*Hint:* use "Theorem A".)

5. Consider the field extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$. Find the Galois group of this extension.

6. Consider the field extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$. Find the Galois group of this extension.

7. Consider the field extension $\mathbb{Q}(\zeta_5)/\mathbb{Q}$, where $\zeta_5 = e^{2\pi i/5}$. Find the Galois group of this extension.

8. Consider the field extension $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}$. Find the Galois group of this extension. (*Hint:* use "Theorem A" twice.)

Definition 2 *Let H be a subgroup of* $\mathrm{Gal}(K/F)$*. The* fixed field of H*, denoted* $\mathrm{Inv}(H)$ *or* $\widehat{I}(H)$ *or* K^H*, consists of all the elements of K that are fixed by all the automorphisms in H. In other words,*

$$\widehat{I}(H) = \{\alpha \in K : \varphi(\alpha) = \alpha \textit{ for all } \varphi \in H\}.$$

9. Show that $\widehat{I}(H)$ is a subfield of K that contains F.

10. If $H_1 \leq H_2$ are subgroups of $\mathrm{Gal}(K/F)$, how are $\widehat{I}(H_1)$ and $\widehat{I}(H_2)$ related?

11. List all the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ for $K = \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\zeta_5), \mathbb{Q}(i, \sqrt{2})$ and find the corresponding fixed fields. (This is perhaps slightly tricky in the third example.)