MAT 347
Irreducibility criteria
January 28, 2020

Let R be a UFD and let F be its field of fractions. Let $f(X) = a_n X^n + \cdots + a_1 X + a_0 \in R[X]$.

The main result

- If $f(X) \in R[X]$ is primitive, then $f(X)$ is irreducible in $R[X] \iff f(X)$ irreducible in $F[X]$.

About roots

- $f(X)$ has a degree 1 factor in $F[X]$ iff it has a root in F.
- Assume $\deg f(X) = 2$ or 3. If f has no roots in F, then it is irreducible in $F[X]$.
- Assume that $\frac{r}{s}$ is a root of $f(X)$ written as a fraction in R in lowest terms.

  Then $r | a_0$ and $s | a_n$.

Reduction

- Let $P \trianglelefteq R$ be a prime ideal. Assume $f(X)$ is monic. Let $\overline{f(X)} \in (R/P)[X]$ be the reduction of $f(X)$.

  If $\overline{f(X)}$ is irreducible in $(R/P)[X]$, then $f(X)$ is irreducible in $R[X]$.

Eisenstein criterion

- Let $P \trianglelefteq R$ be a prime ideal. Assume $f(X)$ is monic; $a_{n-1}, \ldots, a_0 \in P$; and $a_0 \notin P^2$.

  Then $f(X)$ is irreducible in $R[X]$.

Translation

- Let $a \in R$. The map $T_a : R[X] \to R[X], f(X) \mapsto f(X + a)$ is a ring isomorphism.

Exercises

Determine with proof whether each of the following polynomials is irreducible in the given polynomial ring. If they are not, factor them into irreducibles.

1. $f(X) = X^3 + 4X^2 + X - 6$ in $\mathbb{Q}[X]$.
2. $f(X) = X^4 + X^2 + 1$ in $(\mathbb{Z}/2\mathbb{Z})[X]$.
3. $f(X) = X^4 + 1$ in $\mathbb{Z}[X]$.
4. $f(X) = X^5 + 3X^4 + 30X^2 - 9X + 12$ in $\mathbb{Q}[X]$.
5. $f(X) = X^5 + 4X^3 - X + iX + 3 + 3i$ in $\mathbb{Z}[i][X]$.
6. $f(X) = X^3 + 6$ in $(\mathbb{Z}/7\mathbb{Z})[X]$.
7. $f(X, Y) = X^3 + X^2Y + 3XY^2 + 5XY + 2Y$ in $\mathbb{Z}[X, Y]$.
8. $f(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ in $\mathbb{Z}[X]$.

Bonus Exercises

Determine with proof whether each of the following polynomials is irreducible in the given polynomial ring. If they are not, factor them into irreducibles.

9. $f(X) = X^3 + 37X^2 + 300X - 100$ in $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$.
10. $f(X) = X^4 + 4$ in $\mathbb{Z}[X]$.
11. $f(X, Y) = X^3 + Y^3$ in $\mathbb{C}[X, Y]$ and in $\mathbb{Q}[X, Y]$.
12. $f(X, Y, Z) = X^3 + Y^3 + Z^3$ in $\mathbb{C}[X, Y, Z]$.
13. Find all irreducible polynomials of degree 1 and 2 and 3 in $(\mathbb{Z}/2\mathbb{Z})[X]$.
14. Find all irreducible polynomials of degree 1 and 2 in $(\mathbb{Z}/3\mathbb{Z})[X]$ and use that to factor $X^4 + X + 1$ and $X^3 - X$ in $(\mathbb{Z}/3\mathbb{Z})[X]$.

Hints

3. Apply Eisenstein to $f(X + 1)$. 7. Eisenstein. 8. Apply Eisenstein to $f(X + 1)$.

9. Use reduction modulo 3. 11. Use $R = \mathbb{C}[X]$. 12. Use $R = \mathbb{C}[X, Y]$.