# MAT 347
# Classification of finite abelian groups
# November 12, 2019

We want to prove two results:

1. Every finite abelian group is isomorphic to a direct product of cyclic groups.
2. Since different direct products of cylic groups are sometimes isomorphic, we want an easy way to obtain a list of all the abelian groups of order $n$ up to isomorphism, without repetition.

In a way, think of Part 1 as an "existence" result, and Part 2 as a "uniqueness" result.

**I use additive notation for abelian groups throughout this worksheet. I write $Z_a$ for the cyclic group of order $a$.**

## Part 1

1. Prove that every finite abelian group $G$ is isomorphic to the direct product of its Sylow subgroups. (Why does the proof not work for non-abelian groups?) Conclude that it is enough to prove Part 1 for abelian $p$-groups.

   *Hint:* If $P_1, \ldots, P_k$ are Sylow subgroups for the different primes dividing $|G|$, construct a homomorphism $P_1 \times \cdots \times P_k \to G$ and show it is surjective... Or try an inductive argument.

2. Let $G$ be a finite abelian $p$-group. Prove that $G$ has a unique subgroup of order $p$ if and only if $G$ is cyclic.

   *Hint:* For the difficult direction, consider the map $\psi : G \to G$ defined by $\psi(x) = px$ for all $x \in G$ and use induction on $|G|$. Try to apply the induction hypothesis to $\mathrm{im}(\psi)$. It may help to recall Cauchy's Theorem.

3. Let $G$ be a finite abelian $p$-group. Let $A$ be a cyclic subgroup of $G$ of maximal possible order (i.e., generated by an element of maximal order). Prove that $A$ has a *complement*: this means that there exists another subgroup $B \leq G$ such that $A \cap B = 0$ and $A + B = G$ (note that $A + B$ is the additive version of $AB$!).

   *Hint:* Use induction on $|G|$. Deduce from Problem 2 that there exists a subgroup $H$ of order $p$ that is not contained in $A$. Consider the homomorphism $\pi : G \to G/H$ and show that $\pi(A)$ is a cyclic subgroup of maximal possible order of $G/H$...

4. Use Problem 3 to prove Part 1.

## Part 2

5. As a warm-up, complete and prove the following claim:

   Let $a, b$ be positive integers. Then $Z_a \times Z_b \cong Z_{ab}$ iff ...

6. Still as warm-up, show that $Z_{40} \times Z_6 \cong Z_{24} \times Z_{10}$ and that neither of them is isomorphic to $Z_{240}$. (Can you find more ways to write this group as a direct product of two cyclic groups? What about as product of 3 or 4 or more cyclic groups?)

7. Solve Part 2. There are two standard ways to do it. Given a positive integer $n$ we can obtain a list of all abelian groups of order $n$...

   - ... by writing each one as product of as many cyclic groups as possible, or
   - ... by writing each one as product of as few cyclic groups as possible, in some canonical way.

   Either way, you have to prove that every abelian group of order $n$ is isomorphic to one on your list, and that no two different groups on your list are isomorphic to each other.

8. How many abelian groups of order $2^2 \cdot 3^5 \cdot 5^2$ are there up to isomorphism?

## Challenge question

9. *[Putnam 2009 - A5]* Is there a finite abelian group such that the product of the orders of all its elements is $2^{2009}$?