

MAT 347
The action of a group on itself by conjugation
October 16, 2018

Some definitions. Let G be a group.

- Given $g \in G$, the *centralizer* of g in G is the subgroup $C_G(g) := \{a \in G \mid ag = ga\}$.
- Given $S \subseteq G$, the *centralizer* and the *normalizer of S* are the subgroups $C_G(S) := \{a \in G \mid ag = ga \ \forall g \in S\}$ and $N_G(S) := \{a \in G \mid aSa^{-1} = S\}$.
- Two elements $g, h \in G$ are *conjugate* if there exists $a \in G$ such that $h = aga^{-1}$. The conjugacy class of g in G is the set $\text{ccl}_G(g) := \{aga^{-1} \mid a \in G\}$.
- Two subsets $S, T \subseteq G$ are *conjugate* if there exists $a \in G$ such that $T = aSa^{-1}$.

Centers and centralizers

1. Given $S \subseteq G$, what is the relation between S , $C_G(S)$, and $N_G(S)$? In other words: which ones are contained in which ones?

What if $S \leq G$?

2. What is the relation between the cardinalities of the conjugacy class of an element and its centralizer?

Hint: Use the Orbit-Stabilizer theorem.

3. What is the size of the conjugacy class of 3-cycles in S_5 ? What is the size of the centralizer of a 3-cycle in S_5 ? Find all the elements in the centralizer of a 3-cycle in S_5 . Generalize your answer to k -cycle in S_n .
4. Let $S \subseteq G$. What is the relation between the number of conjugates of S and the order of its centralizer/normalizer?
5. Complete the sentence: “The centre of a group $Z(G)$ is the set of elements whose conjugacy class has cardinality ...”
6. Complete the statement of *the Class Equation* and prove it:

Theorem. Let G be a finite group and let g_1, \dots, g_r be [...] Then

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

A lemma

7. Prove or disprove:

- (a) Let G be a group. If $G/Z(G)$ is abelian, then G is abelian.
- (b) Let G be a group. If $G/Z(G)$ is cyclic, then G is cyclic.
- (c) Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.