MAT 347
The symmetric and the alternating groups
October 9, 2018

Recall some definitions.:

- A *permutation* of n elements is an element of the group S_n.
- An m-*cycle* is a permutation that can be written as $(a_1 \cdots a_m)$.
- A *transposition* is a 2-cycle.
- The *cycle type* of a permutation is the set of lengths of the cycles in its decomposition as product of disjoint cycles. For example the cycle type of $(12345)(67)(89)$ in S_{11} is $(5, 2, 2, 1, 1)$.

1. (Products of transpositions)
   (a) Write the permutation (123) as product of transpositions. This can be done in more than one way. Try to write (123) as product of N transpositions, for different values of N. Not all values of N are possible. Which ones are?
   (b) Repeat the same question with the permutation (1234).

   Note: At this point, you can probably make a conjecture for which values of N are not possible, but most likely you won't be able to prove it. For that, we need to introduce some sophistication.

Building the alternating group

Let us fix a positive integer n. Let R be the set of polynomials in the n variables $X_1, \ldots X_n$. We can define an action of S_n on R as follows:

$$\sigma \cdot p(X_1, \ldots, X_n) := p(X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

Make sure you understand what this notation means before continuing. Convince yourself that it is, indeed, an action. You know this action from HW 3 (it will also be relevant again in Galois Theory at the end of the course). We define the following polynomial:

$$\Delta := \prod_{1 \leq i < j \leq n} (X_i - X_j).$$

For example, if $n = 3$, then $\Delta = (X_1 - X_2)(X_1 - X_3)(X_2 - X_3)$.

2. Prove that for every $\sigma \in S_n$ there exists a number $\varepsilon_\sigma \in \{1, -1\}$ such that $\sigma \cdot \Delta = \varepsilon_\sigma \Delta$.
3. Prove that the map $\varepsilon : S_n \to \{1, -1\}$ is a group homomorphism!

We say that a permutation σ is *even* when $\varepsilon_\sigma = 1$ and it is *odd* when $\varepsilon_\sigma = -1$. When we mention the *parity* of a permutation, we are referring to whether it is odd or even. We define A_n to be the set of all even permutations.

4. Complete: "An m-cycle is an even permutation iff m is ...."
5. Go back to the conjecture you made in Question 1. Now you can prove it!
6. Prove that A_n is a normal subgroup of S_n.
7. What is $|A_n|$?

   Hint: Use the first isomorphism theorem.

Conjugacy classes

8. In general, for any G, the *conjugacy class* of $g \in G$ is the orbit of g in the action of G on itself by conjugation. Find a description of the conjugacy classes of S_n.

   Hint: Fix your favourite $\sigma \in S_n$ ($\sigma \neq 1$). Then for various $\tau \in S_n$ compute $\tau\sigma\tau^{-1}$. Can you find a formula for $\tau\sigma\tau^{-1}$? Can you describe the conjugacy class of σ?
9. List all the conjugacy classes of S_5 and the size of each class.

   Hint: You know the sum of the sizes of all the conjugacy classes should be 120, so you can check your final answer.
10. Which of the conjugacy classes in Question 9 are in A_5? Do their sizes add up to the right number?
11. Which of the following sets are generators of S_n?
    (a) The set of all cycles.
    (b) The set of all transpositions.
    (c) The set of all 3-cycles.
    (d) The set $\{(12), (23), (34), \ldots, (n-1, n)\}$.
    (e) The set $\{(12), (13), (14), \ldots, (1n)\}$.

The platonic solids

12. Each one of the five platonic solids has a group of rotations that is isomorphic to either some S_n or some A_n. Find them all (with proof).