MAT 347
Joins and Quotient groups
October 2, 2018

Joins

Let G be a group and let H, K be subgroups. We define $HK := \{hk \mid h \in H, k \in K\}$. We call this the (set-theoretic) product of H and K. In general, this is not a subgroup! It is just a subset of G. Do not confuse this product with the abstract construction of the direct product. We define the *join* of H and K as the smallest subgroup of G containing both H and K. In other words, the join of H and K is $\langle H \cup K \rangle$.

1. Show that $HK \subseteq \langle H \cup K \rangle$.
2. Explore the relation between the following statements (which ones imply which ones)?
   (a) $HK = \langle H \cup K \rangle$.
   (b) $HK \leq G$.
   (c) $HK = KH$. (Notice that this does not mean that the elements of H and the elements of K commute with each other! It only means that HK and KH are the same set.)
3. Find two subgroups H, K of D_8, each of order 2, such that $HK = \langle H \cup K \rangle$.
4. Find two subgroups H, K of D_8, each of order 2, such that $HK \neq \langle H \cup K \rangle$.

Quotient groups

Let G be a group and let $S \subseteq G$. We want to define an equivalence relation in G that will identify all the elements in S, and we want to maintain the group operation. Given $a, b \in G$, we say that $a \sim b$ if there exists $x \in S$ such that $b = ax$. In general, this relation will not be an equivalence relation.

1. Find necessary and sufficient conditions for $\sim$ to be an equivalence relation.

For the rest of this worksheet, let G be a group and let $H \leq G$. We will consider the equivalence relation defined above with $S = H$. Given $a \in G$, the *left coset* of a is the equivalence class of this relation, and we denote it aH. (Why do we use this notation?) The *quotient set* G/H is the set of all equivalence classes. The *index* of H on G, written $|G : H|$ is the number of equivalence classes.

2. Prove that $aH = bH$ iff [there exists $x \in H$ such that $b = ax$] iff $a^{-1}b \in H$
3. In general, what is the cardinality of each coset aH? What is the relation between $|G|$, $|H|$, and $|G : H|$?
4. Consider the group $G = D_8$ and consider the two subgroups $H_1 := \langle s \rangle$ and $H_2 := \langle r \rangle$. For each of them, write the complete list of cosets, and list which elements are in each coset.

Next, we want to try to use the operation on G to define an operation on the set G/H. Given $aH, bH \in G/H$, we can try to define their product by

$$(aH) \star (bH) = (ab)H$$

[*Note:* I am using $\star$ to emphasize that I am defining a new operation. As soon as we make sure this operation works and there is no ambiguity, we will drop the $\star$.]

5. In general, the operation $\star$ is not well-defined. Going back to the example in Question 4, show that with one of those subgroups, the operation is well-defined, but with the other subgroup, the operation is not well-defined.