MAT 347
An example of the FTGT
March 19, 2019

The Fundamental Theorem of Galois Theory

Definition. A finite field extension K/F is *Galois* if it is normal and separable.

Theorem. *Let K/F be a finite Galois extension. Let $G = \mathrm{Gal}(K/F)$. Consider the maps $\widehat{I}$ and $\widehat{G}$ from Section 4 of Alfonso's notes.*

(1) The maps $\widehat{I}$ and $\widehat{G}$ are inverses of each other. In other words:

- $\widehat{G}(\widehat{I}(H)) = \mathrm{Gal}(K/\operatorname{Inv}(H)) = H$ *for every subgroup* $H \leq G$.
- $\widehat{I}(\widehat{G}(M)) = \operatorname{Inv}(\mathrm{Gal}(K/M)) = M$ *for every intermediate field* $F \subseteq M \subseteq K$.

(2) Let $H \leq G$ and let $M = \widehat{I}(H)$, so that $H = \widehat{G}(M)$.
Then $|H| = [K : M]$. In particular $|G| = [K : F]$.
Equivalently, $(G : H) = [M : F]$.

(3) Under the same conditions as in Part (2), K/M is always Galois. In addition, TFAE:

- *M/F is Galois.*
- *M/F is a normal field extension.*
- *H is a normal subgroup of G.*

In that case, $\mathrm{Gal}(M/F) \cong \mathrm{Gal}(K/F)/\mathrm{Gal}(K/M)$.

An Example

Let $f(X) = X^4 - 2$. Let $K \subset \mathbb{C}$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$.

1. Find all roots of $f(X)$ in $\mathbb{C}$.
2. Find a (nice) set of two elements that generate the field extension $K/\mathbb{Q}$.
3. Calculate $[K : \mathbb{Q}]$.

4. Find a basis for K as a $\mathbb{Q}$-vector space.
5. List all the elements of G by showing how they act on a set of generators.
6. Determine the group G up to isomorphism.
7. Find all intermediate fields of $K/\mathbb{Q}$.
8. Which intermediate fields are normal over $\mathbb{Q}$? Each one of them has to be the splitting field of some polynomial. Find these polynomials.
9. Find a primitive element for the field extension $K/\mathbb{Q}$, i.e. an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. (*Hint:* find an α such that $\widehat{G}(\mathbb{Q}(\alpha)) = 1 \ldots$)