MAT 347
Constructions with straightedge and compass
February 12, 2019

General constructions

For centuries, mathematicians searched for methods to solve geometric constructions using only straightedge and compass. It never occurred to the ancient Greeks that some of these constructions were impossible.

Definition 1 *Let P_0 be a set of points in the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$. The two basic constructions are*

> Operation 1 (straightedge)*: draw a line through any two points of P_0.*
>
> Operation 2 (compass)*: draw a circle centered at any point of P_0 and with radius equal to the distance between a pair of points in P_0.*

The points of intersection of any lines and circles drawn using Operations 1 and 2 are said to be constructible from P_0 in one step*. Given $r \in \mathbb{R}^2$, we say that r is* constructible from P_0 *if there exist points $r_1, r_2, \ldots, r_n = r$ such that r_i is constructible in one step from*

$$P_0 \cup \{r_1, \ldots, r_{i-1}\}$$

for $i = 1, \ldots, n$.

1. Show that using Operations 1 and 2 we can do the following constructions:
   (a) Bisect a given segment.
   (b) Draw the bisector of a given angle.
   (c) Given a line l and a point P, draw a line through P perpendicular to l
   (d) Given a line l and a point P, draw a line through P parallel to l.

2. Let $z \in \mathbb{C}$. Show that the following are equivalent:
   (a) z can be constructed from $\{0, 1\}$.
   (b) We can construct segments with length $\operatorname{Re} z$ and $\operatorname{Im} z$ starting from a single segment of length 1 by using Operations 1 and 2.

3. Here are three classical construction problems. Translate each one of them into a statement of the form "the point z can be constructed from the set of points P_0".

   (a) Squaring the circle: given a circle, construct a square that has the same area.

   (b) Doubling the cube: given a cube, construct a cube with volume twice as large.

   (c) Trisecting an arbitrary angle.

Definition 2 *We denote by Δ_2 the set of all $z \in \mathbb{C}$ that can be constructed from* $\{0, 1\}$.

We will now show that Δ_2 is a subfield of $\mathbb{C}$.

4. Show that Δ_2 is a subgroup (under addition) of $\mathbb{C}$.

5. Show that if $\alpha, \beta \in \Delta_2$, then $\alpha\beta \in \Delta_2$.

   (*Hint:* if it helps, you can reduce to real α, β using Question 2.)

6. Show that if $\alpha \in \Delta_2 - \{0\}$, then $\alpha^{-1} \in \Delta_2$. Conclude that Δ_2 is a subfield of $\mathbb{C}$.

Now we have to prove that Δ_2 is not all of $\mathbb{C}$, or at least not all of the algebraic numbers over $\mathbb{Q}$. The way we will do this is by examining what happens when we do one construction. Note first that $\mathbb{Q}(i) \subseteq \Delta_2$.

7. Let F be a field with $\mathbb{Q}(i) \subseteq F \subseteq \mathbb{C}$ such that $\overline{F} = F$, where $\overline{F} = \{\overline{z} : z \in F\}$. Let $z \in \mathbb{C}$. Assume that z can be constructed from a set of points in F in just one step. Prove that $[F(z) : F] = 1$ or 2 and that $\overline{F(z)} = F(z)$.

   (*Hint:* first show that $z \in F$ iff both $\operatorname{Re} z$ and $\operatorname{Im} z$ are in F. Then you can use equations for lines and circles in $\mathbb{R}^2$.)

8. Let $z \in \mathbb{C}$. Assume that $z \in \Delta_2$. Show that there exist fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$ such that $z \in F_n$ and $[F_i : F_{i-1}] = 2$ for every i.

9. Let $z \in \mathbb{C}$. If $z \in \Delta_2$ show that $[\mathbb{Q}(z) : \mathbb{Q}] = \deg m_{z,\mathbb{Q}}(X)$ is a power of 2.

10. Show the following results.

    (a) It is impossible to square the circle.

    (b) It is impossible to double the cube.

    (c) It is impossible to trisect an arbitrary angle.

    (*Hint for (c):* remember/look up the cosine triple angle formula and pick a convenient angle.)