MAT 347
Subrings, ideals, and ring homomorphisms
November 27, 2018

A lot of the concepts we want to define for rings, and a lot of the theorems we want to prove for rings, are entirely analogous to their counterparts for groups. Your task is to think of all the group theory you know and find out what generalizes and how.

Definitions and warm-up

Some definitions. A *ring* is a set R together with two binary operations $+, \times$ satisfying:

(i) $(R, +)$ is an abelian group;

(ii) multiplication is associative;

(iii) the distributive laws hold: $a \times (b+c) = (a \times b) + (a \times c)$, $(b+c) \times a = (b \times a) + (c \times a)$;

(iv) there is a multiplicative identity 1: $a \times 1 = a = 1 \times a$.

Usually we write ab for $a \times b$. Some easy consequences:

- 1 is unique;
- $a0 = 0 = 0a$;
- $(-a)b = -(ab) = a(-b)$;

Warning: the book does not assume that a ring has a multiplicative identity 1!

A ring is *commutative* if multiplication $\times$ is commutative.

1. Find all rings in which $0 = 1$.

Subrings

2. Let A be a ring. Let $R \subseteq A$ be a subset. We say that R is a *subring* of A, and we write $R \leq A$, when R is also a ring with the same operations as A and the same identity 1. Write down the criterion for quickly checking when a subset of a ring is a subring (analogous to the criterion for quickly checking when a subset of a group is a subgroup).

3. Find all subrings of $\mathbb{Z}$ and at least 3 subrings of $\mathbb{Q}$ (there are two easy ones...).

Quotient rings and ideals

4. Let $(A, +, \cdot)$ be a ring and let $I \leq A$ be a subgroup under addition. We want to define the quotient A/I. For every $a \in A$ we define

$$a + I := \{a + r \mid r \in I\}.$$

We also define the quotient set

$$A/I := \{a + I \mid a \in A\}.$$

We want to define operations on the quotient A/I. Specifically, we want to define

$$(a + I) + (b + I) := (a + b) + I;$$
$$(a + I)(b + I) := ab + I.$$

We say that I is an *ideal* when these operations are well-defined. We write $I \lhd A$ or $I \unlhd A$. Find a simple algebraic characterization of ideal, just like the characterization of normal subgroups as "closed under conjugation".

5. Assume that I is an ideal. What else do we need to require for A/I to be a ring? If A is commutative, does it follow that A/I is commutative?

6. Find all the ideals of $\mathbb{Q}$. In which other rings would you get the same result as for $\mathbb{Q}$?

7. Find all the ideals of $\mathbb{Z}$.

Ring homomorphisms

8. Define ring homomorphism and ring isomorphism.

9. Define the kernel of a ring homomorphism.

10. The kernel of a group homomorphism is always a normal subgroup. State and prove the corresponding result for rings.

11. A group homomorphism is injective if and only if its kernel is trivial. State and prove the corresponding result for rings.

12. State and prove the First Isomorphism Theorem for rings. (Imitate the corresponding theorem for groups.)