MAT 347
Classification of finite abelian groups
November 13, 2018

We want to prove two results:

1. Every finite abelian group is isomorphic to a direct product of cyclic groups.
2. Since different direct products of cylic groups are sometimes isomorphic, we want an easy way to obtain a list of all the abelian groups of order n up to isomorphism, without repetition.

In a way, think of Part 1 as an "existence" result, and Part 2 as a "uniqueness" result.

I use additive notation for abelian groups throughout this worksheet. I write Z_a for the cyclic group of order a.

Part 1

1. Prove that every finite abelian group G is isomorphic to the direct product of its Sylow subgroups. (Why does the proof not work for non-abelian groups?) Conclude that it is enough to prove Part 1 for abelian p-groups.

   Hint: If $P_1, \ldots, P_k$ are Sylow subgroups for the different primes dividing $|G|$, consider a homomorphism $P_1 \times \cdots \times P_k \to G$ and show it is surjective... Or try an inductive argument.

2. Let G be a finite abelian p-group. Prove that G has a unique subgroup of order p if and only if G is cyclic.

   Hint: For the difficult direction, consider the map $\psi : G \to G$ defined by $\psi(x) = px$ for all $x \in G$ and use induction on $|G|$. Try to apply the induction hypothesis to $\text{im}(\psi)$. It may help to recall Cauchy's Theorem.

3. Let G be a finite abelian p-group. Let A be a cyclic subgroup of G of maximal possible order (i.e., generated by an element of maximal order). Prove that A has a *complement*: this means that there exists another subgroup $B \leq G$ such that $A \cap B = 0$ and $A + B = G$ (note that $A + B$ is the additive version of AB!).

   Hint: Use induction on $|G|$. Deduce from Problem 2 that there exists a subgroup H of order p that is not contained in A. Consider the homomorphism $\pi : G \to G/H$ and show that $\pi(A)$ is a cyclic subgroup of maximal possible order of G/H...

4. Use Problem 3 to prove Part 1.

Part 2

5. As a warm-up, complete and prove the following claim:

   Let a, b be positive integers. Then $Z_a \times Z_b \cong Z_{ab}$ iff ...

6. Still as warm-up, show that $Z_{20} \times Z_6 \cong Z_{12} \times Z_{10}$ and that neither of them is isomorphic to Z_{120}. (Can you find more ways to write this group as a direct product of two cyclic groups? What about as product of 3 or 4 or more cyclic groups?)

7. Solve Part 2. There are two standard ways to do it. Given a positive integer n we can obtain a list of all abelian groups of order n...

   - ... by writing each one as product of as many cyclic groups as possible, or
   - ... by writing each one as product of as few cyclic groups as possible, in some canonical way.

   Either way, you have to prove that every abelian group of order n is isomorphic to one on your list, and that no two different groups on your list are isomorphic to each other.

8. How many abelian groups of order $2^5 \cdot 3^2 \cdot 5^2$ are there up to isomorphism?

Challenge question

9. *[Putnam 2009 - A5]* Is there a finite abelian group such that the product of the orders of all its elements is 2^{2009}?