MAT 1200/415, Algebraic Number Theory, Fall 2018
Homework 5, due on Friday December 14
Florian Herzig

1. Milne, Appendix B, problem 6.

2. Cassels, chapter 7, exercises 5 (only (i)–(iii)), 13 (not 13 bis), 18.

3. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_{p^r \cdot s})$, where $p \nmid s$. Show that $\mathbb{Q}(\zeta_s)$ is the largest subfield of K in which p is unramified.

4. Consider $K = \mathbb{Q}(\alpha)$, where α is a root of the irreducible polynomial $x^3 + x^2 - 6x - 7$, which has discriminant $361 = 19^2$. Note that $K/\mathbb{Q}$ is a cylic extension, since the discriminant is a square. The Kronecker-Weber theorem implies that K is contained in a cyclotomic field $\mathbb{Q}(\zeta_n)$.

   (a) Use the previous problem to show that K is contained in $\mathbb{Q}(\zeta_{19^r})$ for some r.

   (b) Then prove that K is contained in $\mathbb{Q}(\zeta_{19})$.

   (c) Determine the set of primes that split completely in K.

   (In these two problems you will need to know Euler's ϕ-function.)

5. Consider $K = \mathbb{Q}(\sqrt[3]{10})$. Recall that in Homework 3 you showed that $(3) = \mathfrak{p}\mathfrak{q}^2$ in $\mathcal{O}_K$, where $\mathfrak{p} \neq \mathfrak{q}$ are prime ideals, by using a clever argument relying on the knowledge of $\mathcal{O}_K$. The goal of this exercise is to give two alternative proofs of this that avoid knowing $\mathcal{O}_K$.

   (a) Use Newton's method to show that $f(x) = x^3 - 10$ has precisely one root α in $\mathbb{Q}_3$. Use this to deduce the claim. You may use that 3 ramifies in K from Homework 3. Also, determine α modulo $3^7\mathbb{Z}_3$.

   (b) Alternatively, consider the Newton polygon of $f(x+1)$ to deduce the claim. (This time do not assume you already know that 3 ramifies in K.)

   Note that it's natural to consider $f(x+1)$, since any root of $f(x)$ in $\mathbb{Z}_3$ has to be $\equiv 1 \pmod 3$.

6. Consider the polynomial $f(x) = 5!(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}) \in \mathbb{Z}[x]$.

   (a) Let p be 2, 3, or 5. By using the Newton polygon, determine the degrees of the irreducible factors of $f(x)$ over $\mathbb{Q}_p$. Deduce from this that $f(x)$ is irreducible over $\mathbb{Q}$.

   (b) Using the previous part, show that the Galois group of $f(x)$ is either A_5 or S_5. (Consider bounding e's and f's for the Galois closure.)

In fact, the Galois group is S_5, as can be checked using the discriminant.