MAT 1200/415, Algebraic Number Theory, Fall 2018
Homework 4, due on Friday November 16

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1. Marcus, Number fields, Chapter 5, Problem 33. Then determine the unit group of $\mathcal{O}_K$ where $K = \mathbb{Q}(\sqrt{d})$ for d equal to 5, 14.

2. Show that the class number of $\mathbb{Q}(\gamma)$ is 1, where $\gamma^3 + \gamma + 1 = 0$.

3. The goal of this exercise is to show that $K = \mathbb{Q}(\sqrt[3]{10})$ has class number 1. Let $\alpha = \sqrt[3]{10}$ and $\beta = (\alpha - 1)^2/3$. Recall that you have already worked out a few things about this field on previous homeworks: $\mathcal{O}_K$ has basis $1, \alpha, \beta$, and you should have seen the following prime ideal factorisations: $2 = (2, \alpha)^3$, $3 = (3, \beta)(3, \beta + 1)^2$, $5 = (5, \alpha)^3$, 7 is prime, $11 = (11, \alpha + 1)(11, \alpha^2 - \alpha + 1)$, 13 is prime.

   (a) Show that K has class number 1, provided that all the prime ideals above are principal.

   You don't even need all of them, if you compute the Minkowski bound correctly!

   (b) Use the identity $N_{K/\mathbb{Q}}(a + b\alpha + c\alpha^2) = a^3 + 10b^3 + 100c^3 - 30abc$ to show that the prime ideals dividing 2, 5, or 11 are principal. (It might help in some cases to use that a product/ratio of principal ideals is principal...)

   (c) Compute $N_{K/\mathbb{Q}}(\beta+1)$ and perhaps $N_{K/\mathbb{Q}}(\beta)$ to finish the problem.

   (d) Use an equality of two ideals of small norm to find a unit in $\mathcal{O}_K^\times$ not equal to ± 1. (See the short section in Milne entitled "Finding a system of fundamental units".)

4. Read Section 5 of `http://www.math.uni-bonn.de/people/tian/ANT.pdf` (as much as you need), considering only the case of negative discriminant (which simplifies things, as the "narrow class group" is just the usual class group). Suppose $K = \mathbb{Q}(\sqrt{-199})$. Find the class number of K by determining all reduced (primitive) binary quadratic forms of discriminant -199. (Optional: what is the class group?)