MAT 1200/415, Algebraic Number Theory, Fall 2018
Homework 3, due on Friday October 26

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1. Marcus, Number fields, Chapter 3, Problem 12. (Do not use Kummer-Dedekind in this problem! Optional: can you see directly why the map in (c) is an isomorphism?)

2. Determine exactly which prime numbers p split, ramify, and remain inert in $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{6})$ or $\mathbb{Q}(\sqrt{-11})$. Your answer should be expressed in terms of congruence conditions on p.

3. Consider $K = \mathbb{Q}(\sqrt[3]{10})$.

   (a) Find all prime numbers $p \neq 3$ that ramify in $\mathcal{O}_K$ and factor $p\mathcal{O}_K$ into prime ideals in each case.

   (b) Suppose that $p \equiv 1 \pmod 3$ is unramified in $\mathcal{O}_K$. What are the possible splitting behaviours of p in $\mathcal{O}_K$ (i.e. how many primes does $p\mathcal{O}_K$ factor into, and what are their residue degrees f_i)? Give an example of a prime p and its factorisation in $\mathcal{O}_K$ for each possible splitting behaviour.

   (c) Now repeat the same question for $p \equiv 2 \pmod 3$ unramified in $\mathcal{O}_K$.

   (d) Factor the prime 3 in $\mathcal{O}_K$, by following Marcus Chapter 3, Problem 26(d). (This is very special to this field… Later we will see a nice way to determine the splitting behaviour of (3) using p-adic techniques.)

   (Hint: remember Homework 2 for $\mathcal{O}_K$, as well as Kummer-Dedekind!)

4. Recall that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain that is not a PID. Find a very indirect proof that there are infinitely many prime numbers in $\mathbb{Z}$ by showing that otherwise $\mathbb{Z}[\sqrt{-5}]$ would be a PID.

5. Suppose that I is any fractional ideal in a Dedekind domain A.

   (a) Show that there is a surjection of A-modules $\pi : A^2 \to I$ (where $A^2 = A \oplus A$).

(b) Find a map of A-modules $s : I \to A^2$ such that $\pi \circ s = 1_I$, the identity function $I \to I$.

(c) Deduce that there is an A-module J such that $A^2 \cong I \oplus J$. (Hint: try $J = \ker(\pi)$.)

(d) Give an example of an I that is not free as an A-module. (Remark: this is an example of a projective module that is not free.)

(Hint: recall that fractional ideals in Dedekind domains are invertible and can be generated by two elements. Do not use the classification of finitely generated modules over Dedekind domains that I mentioned in class!)