MAT 1200/415, Algebraic Number Theory, Fall 2018
Homework 2, due on Friday October 12
Florian Herzig

1. Marcus, Number fields, Chapter 2, read his Theorem 13, then do Problems 40, 41 for $m = 10$ only (41(a) should say $\operatorname{disc}(\alpha) = -27m^2$!).

2. Suppose that $A \subset B$ are domains and that S is a multiplicative subset of A. If C is the integral closure of A in B, show that $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$. (This will be useful below!)

3. We are about to show that for any number field K, $\mathcal{O}_K$ is a Dedekind domain and hence every nonzero ideal is a product of prime ideals (uniquely, up to ordering). Recall that $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$. Explain why these two statements don't contradict each other. (Hint: first show $(2) = (2, 1+\sqrt{-5})(2, 1-\sqrt{-5})$.)

4. Consider the extension of number fields $L = \mathbb{Q}(\sqrt{-6}, \sqrt{-3})$ over $K = \mathbb{Q}(\sqrt{-6})$. In this case Keith Conrad's notes (posted on the course website) show that $\mathcal{O}_L$ isn't free as $\mathcal{O}_K$-module, so our definition of discriminants from class doesn't work. You may assume the following:

(Fact 1) There is an *ideal* $\operatorname{disc}(\mathcal{O}_L/\mathcal{O}_K)$ of $\mathcal{O}_K$ such that whenever S is a multiplicative subset of $\mathcal{O}_K$ such that $0 \notin S$ and $S^{-1}\mathcal{O}_L$ is free as $S^{-1}\mathcal{O}_K$-module, we have $S^{-1}\operatorname{disc}(\mathcal{O}_L/\mathcal{O}_K)$ is the principal ideal of $S^{-1}\mathcal{O}_K$ that we defined in class (i.e. it's generated by the element $\operatorname{disc}(S^{-1}\mathcal{O}_L/S^{-1}\mathcal{O}_K) \in \mathcal{O}_K$, which was well-defined up to squares of units).

(Fact 2) $\mathcal{O}_L$ has $\mathbb{Z}$-basis $1, \frac{1+\sqrt{-3}}{2}, \sqrt{-6}, \frac{\sqrt{-6}+\sqrt{2}}{2}$ (see Marcus, Problem 2/42).

Using this, show:

(a) If we take $S = \{2^n : n \geq 0\}$, then $S^{-1}\mathcal{O}_L$ is free as $S^{-1}\mathcal{O}_K$-module with basis $1, \frac{\sqrt{-6}+\sqrt{2}}{2}$. Deduce that $S^{-1}\operatorname{disc}(\mathcal{O}_L/\mathcal{O}_K) = (1)$.

(b) If we take $T = \{3^n : n \geq 0\}$, then $T^{-1}\mathcal{O}_L$ is free as $T^{-1}\mathcal{O}_K$-module with basis $1, \frac{1+\sqrt{-3}}{2}$. Deduce that $T^{-1}\operatorname{disc}(\mathcal{O}_L/\mathcal{O}_K) = (1)$.

(c) Use the previous parts to show that $\operatorname{disc}(\mathcal{O}_L/\mathcal{O}_K) = (1)$.

(This problem looks more complicated than it is.)