

MAT 1100, Algebra I, Fall 2015
Homework 5, due on Wednesday December 9
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1. Suppose R is a domain and M an R -module. Recall that $M_{\text{tor}} = \{x \in M : rx = 0 \text{ for some } r \in R - \{0\}\}$, a submodule of M . We say that M is *torsion* if $M_{\text{tor}} = M$.
 - (a) Let $\text{Ann}_R(M) := \{r \in R : rm = 0 \forall m \in M\}$, the *annihilator* of M . Show that $\text{Ann}_R(M)$ is an ideal of R .
 - (b) Suppose that I, J are ideals of R such that $R/I \cong R/J$ as R -modules. Show that $I = J$. (Hint: consider annihilators.)
 - (c) If M is a finitely generated torsion R -module show that $\text{Ann}_R(M) \neq 0$.
 - (d) Give an example of a domain R and a *torsion* R -module M such that $\text{Ann}_R(M) = 0$.
2.
 - (a) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. (Hint: construct maps in both directions.)
 - (b) Find $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.
 - (c) Consider $V := \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2$, and let e_1, e_2 be the standard basis of \mathbb{R}^2 . Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V$ isn't a pure tensor, i.e. is not of the form $v \otimes w$ for some $v, w \in \mathbb{R}^2$.
3. Suppose R is a commutative ring, M an R -module, and $I, J \triangleleft R$ ideals.
 - (a) Show that $R/I \otimes_R M \cong M/IM$, where IM is the submodule of M generated by the elements rm ($r \in I, m \in M$). (Hint: construct maps in both directions.)
 - (b) Deduce that $R/I \otimes_R R/J \cong R/(I + J)$.
 - (c) Let $f(x) := (x^2 + 1)^5(x^4 - 1)$, $g(x) := (x - i)^2(x^2 - 1)^3$. Express the $\mathbb{C}[x]$ -module $\mathbb{C}[x]/(f(x)) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(g(x))$ in both canonical forms that we discussed in class (cf. Theorems 79 and 80).
 - (d) Now redo part (c) for the module $\mathbb{C}[x]/(f(x)) \oplus \mathbb{C}[x]/(g(x))$.
4.
 - (a) Find all abelian groups of order $720 = 2^4 \cdot 3^2 \cdot 5$, up to isomorphism. How many are there?

- (b) Show that any abelian group of order 2015 is cyclic.
5. (a) Suppose K is a field. For any monic irreducible polynomial $f(x) \in K[x]$ of degree $d > 0$ and any integer $e > 0$ determine the possible $K[x]$ -linear homomorphisms $K[x]/(f(x)^e) \rightarrow K[x]/(f(x)^e)$, and describe which of them are invertible. When $K = \mathbb{F}_p$, how many are there of each kind?
- (b) Use part (a) to determine the size of the centraliser and of the conjugacy class for the following matrices in $\mathrm{GL}_3(\mathbb{F}_p)$:

$$\begin{pmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{pmatrix}, C_{g(x)},$$

where α, β, γ are pairwise distinct elements of \mathbb{F}_p^\times and $g(x) \in \mathbb{F}_p[x]$ is a monic irreducible polynomial of degree 3. (You may use that $|\mathrm{GL}_3(\mathbb{F}_p)| = (p^3 - 1)(p^3 - p)(p^3 - p^2)$.)

- (c) How many conjugacy classes are there in $\mathrm{GL}_2(\mathbb{F}_p)$? (You may use that there are precisely $(p^2 - p)/2$ monic irreducible quadratic polynomials in $\mathbb{F}_p[x]$.)
6. Suppose K is a field.
- (a) Show that any companion matrix $C_{g(x)} \in M_n(K)$ is conjugate to its transpose matrix (by an element of $\mathrm{GL}_n(K)$). (Hint: it might be easiest to use the minimal or characteristic polynomials to compute the invariant factors of the transpose matrix.)
- (b) Use part (a) to show that any matrix $A \in M_n(K)$ is conjugate to its transpose matrix tA (by an element of $\mathrm{GL}_n(K)$).