MAT 1100, Algebra I, Fall 2016
Homework 4, due on Tuesday November 22

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All rings in problems 2–6 are commutative.

1. (a) Show that the map $\mathbb{C} \to M_2(\mathbb{R})$ sending $a + bi$ to $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ is a ring homomorphism.

   (b) Suppose that S is a ring, $R = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in S\}$ (a subring of $M_2(S)$), and that $I = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) : a, b \in S\}$. Show that I is an ideal of R by finding a ring homomorphism from R to some other ring that has kernel I.

   (c) Show that x^2+1 is irreducible in $\mathbb{F}_3[x]$. Deduce that $\mathbb{F}_3[x]/(x^2+1)$ is a field. Find its order $N = |\mathbb{F}_3[x]/(x^2+1)|$.

   (d) Using the notation of (iii), show that $\mathbb{F}_3[x]/(x^2+1) \not\cong \mathbb{Z}/N\mathbb{Z}$ as rings.

2. (a) Suppose $I \lhd R$ is an ideal. Suppose that $\overline{J} \lhd R/I$ corresponds to $J \lhd R$ in the correspondence theorem. Show that $\overline{J}$ is a prime (resp. maximal) ideal in R/I iff J is a prime (resp. maximal) ideal in R.

   (b) Determine all prime ideals in $\mathbb{R}[x]/(x^3 - 3x - 2)$. Which of them are maximal?

3. The ring $R = \mathbb{Z}[\sqrt{-2}]$ is a UFD (you can assume this as a fact). Consider the norm function $N : R \to \mathbb{Z}_{\geq 0}$, $z \mapsto |z|^2$ as in class.

   (a) Determine the units $R^\times$.

   (b) Factor the following elements into primes in the ring R: 2, 5, 19.

   (c) Show that a prime number p is prime in R iff $x^2 + 2y^2 = p$ has no solutions with $x, y \in \mathbb{Z}$.

   (d) Find the gcd of $-2 + \sqrt{-2}$ and 3 in R.

4. Suppose that $\{P_i : i \in I\}$ is a chain of prime ideals of a ring R (for some set I). Prove that $\bigcap_{i \in I} P_i$ is a prime ideal. Deduce that any prime ideal of R contains a minimal prime ideal. (As usual, a minimal prime

ideal is one that does not contain any strictly smaller prime ideal. For example, in any domain, the ideal (0) is a minimal prime ideal, in fact it is the unique one.)

5. (a) If $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_n a_0 \neq 0$ has a root $\alpha \in \mathbb{Q}$, show that $\alpha = \frac{r}{s}$ with integers r, s such that $r | a_0$ and $s | a_n$. (Hint: use a version of Gauss' lemma to deduce that f has a linear factor in $\mathbb{Z}[x]$.)

   (b) Show that $x^4 + 5x^3 + 3x^2 - x - 1$ is irreducible in $\mathbb{Q}[x]$. (Hint: use Gauss' lemma. To rule out quadratic factors, write it as a product of unknown quadratic polynomials in $\mathbb{Z}[x] \ldots$)

   (c) Is $x^4 + 4x^2 - x + 6$ irreducible in $\mathbb{Q}[x]$?

6. Suppose R is a UFD. Define the *content* $c(f)$ for $f = a_n x_n + \cdots + a_1 x + a_0 \in R[x]$, $f \neq 0$ to be $c(f) = \gcd(a_n, \ldots, a_0)$.

   (a) Show that for f as above we can write $f = b f_0$, where $b \in R$ and $f_0 \in R[x]$ is primitive. Show that in this decomposition, b and f_0 are unique up to associates.

   (b) Show that the product of two primitive polynomials is primitive. (Hint: a result from class about prime elements is useful.)

   (c) Deduce that $c(fg) = c(f)c(g)$ for f, $g \in R[x] - \{0\}$.